

A Logician's View of Graph Polynomials

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Abstract

Graph polynomials are graph parameters invariant under graph isomorphisms which take values in a polynomial ring with a fixed finite number of indeterminates. We study graph polynomials from a model theoretic point of view. In this paper we distinguish between the graph theoretic (semantic) and the algebraic (syntactic) meaning of graph polynomials. We discuss how to represent and compare graph polynomials by their distinctive power. We introduce the class of graph polynomials definable using Second Order Logic which comprises virtually all examples of graph polynomials with a fixed finite set of indeterminates. Finally we show that the location of zeros and stability of graph polynomials is not a semantic property. The paper emphasizes a model theoretic view and gives a unified exposition of classical results in algebraic combinatorics together with new and some of our previously obtained results scattered in the graph theoretic literature.

Keywords: Graph polynomials, Second Order Logic, Definability, Chromatic Polynomials, Generating functions

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1. Introduction

This paper gives a logician's view of some aspects of graph polynomials. A short version was given as an invited lecture by the first author at WOLLIC 2016, [MR16].

A graph $G = (V(G), E(G))$ is given by the set of vertices $V(G)$ and a symmetric edge-relation $E(G)$. We denote by $n(G)$ the number of vertices, by $m(G)$ the number of edges, by $k(G)$ the number of connected components of a graph G , and by \mathcal{G} the class of finite graphs.

Graph polynomials are graph invariants with values in a polynomial ring \mathcal{R} , usually $\mathbb{Z}[\mathbf{X}]$ with $\mathbf{X} = (X_1, \dots, X_\ell)$. Let $\mathbf{P}(G; \mathbf{X})$ be a graph polynomial of the form

$$\mathbf{P}(G; \mathbf{X}) = \sum_{i_1, \dots, i_\ell=0}^{d(G)} c_{i_1, \dots, i_\ell}(G) X_1^{i_1} \cdot \dots \cdot X_\ell^{i_\ell}$$

where $\mathbf{X} = (X_1, \dots, X_\ell)$, $d(G)$ is a graph parameter with non-negative integers as its values, and

$$c_{i_1, \dots, i_\ell}(G) : i_1, \dots, i_\ell \leq d(G)$$

are integer valued graph parameters.

Definition 1.1. A graph polynomial \mathbf{P} is computable if

- (i) \mathbf{P} is a Turing computable function, and additionally,
- (ii) the range of \mathbf{P} , the set

$$\{p(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}] : \text{there is a graph } G \text{ with } \mathbf{P}(G; \mathbf{X}) = p(\mathbf{X})\}$$

is Turing decidable.

The second condition is needed to make Theorem 2.2 work.

Graph polynomials have been studied for the last hundred years, since G. Birkhoff introduced his chromatic polynomial in [Bir12]. This was generalized by H. Whitney in the 1930ties, [Whi32] and led to the Tutte polynomial, also called the *dichromate* or the *Tutte-Whitney polynomial*. For a history see [Far07]. Motivated by questions in theoretical chemistry, the characteristic polynomial and the matching polynomial of graphs were introduced, and studied intensively, [Hos71, HL72, Tri92, Bal93, Bal95, Hos02]. In the last 30 years many more graph polynomials appeared in the literature. The abundance of graph polynomials which appear in the more recent literature leads to various questions:

- How to compare graph polynomials?
- What kind of information may be extracted from a graph polynomial about its underlying graph?
- Are there any normal forms of graph polynomials?

Ten years ago B. Zilber and the first author have discovered a connection between model theory and graph polynomials, [MZ06, KMZ11]. In [MRB14] we introduced the distinction between syntactic and semantic properties of graph polynomials. Our discussion in these and subsequent papers, was mostly addressed the graph theory community. This paper is written for the logically minded and is a continuation of our analysis of notions used in the literature on graph polynomials. In particular, we were bothered by the question whether the location of the roots of univariate graph polynomials is a combinatorially meaningful statement about its underlying graph. The notion of “combinatorially meaningful” is made precise by our definition of a *semantic (aka graph theoretic) property of graph polynomials*. In [MRB14] we have given our analysis of this question for the graph theory audience. Here we want to stress the *logical and foundational aspects* of this analysis, and extend the results of [MRB14] to multivariate graph polynomials.

1.1. Why SOL-definability

There are too many graph polynomials if they are merely defined as graph invariants with values in a polynomial ring. We can impose more restrictions by imposing *computability* and *definability* requirements. Imposing complexity theoretic restrictions poses some serious problems, and is studied in [MKR13]. However, it is not the subject of this paper.

It is more natural to impose definability restrictions. In [Mak04, AGM10, KMZ11, GKM12] the class of graph polynomials definable in Second Order Logic SOL, is studied, which requires that $d(G)$ and $c_1(G) = c(G; 1)$, $1 = (i_1, \dots, i_\ell)$, are, even uniformly, definable in SOL. With very few exceptions, the graph polynomials studied in the literature are SOL-definable⁴. We assume the reader is familiar with Finite Model Theory, cf. [EF95, Lib04]. The finite model theory of graph polynomials was developed in [Mak08, Kot12, KMZ11]. For the convenience of the reader it will be summarized in Section 5.

Requiring that the graph polynomials are SOL-definable also guarantees that their coefficients are the result of counting *combinatorially meaningful* SOL-definable configurations in the underlying graph.

1.2. Why study graph polynomials?

The first graph polynomial, the chromatic polynomial, was introduced in 1912 by G. Birkhoff to study the Four Color Conjecture, [Bir12]. The emergence of the Tutte polynomial can be seen as an attempt to generalize the chromatic polynomial, cf. [Tut54, Bol98, EMC11]. The characteristic polynomial and the matching polynomial were introduced with applications from chemistry in mind, cf. [Tri92, Bal93, Bal95, CDS95, BH12]. Physicists study various partition functions in statistical mechanics, in percolation theory and

⁴ Many are even definable in Monadic Second Order Logic MSOL, [Mak08]. The exceptions are in [NW99]. The algorithmic advantages of MSOL-definability, [CMR01] are of no importance in this paper.

in the study of phase transitions, cf. [NW04]. It turns out that many partition functions are incarnations of the Tutte polynomial. Another incarnation of the Tutte polynomial is the Jones polynomial in Knot Theory, [Jae88] and again [Bol98]. The various incarnations of the Tutte polynomial have triggered an interest in other graph polynomials. These graph polynomials are studied for various reasons:

- Graph polynomials can be used to distinguish non-isomorphic graphs. A graph polynomial is *complete* if it distinguishes all non-isomorphic graphs. The quest for a complete graph polynomial which is also easy to compute failed so far for two reasons. Either there were too many non-isomorphic graphs which could not be distinguished, and/or the proposed graph polynomial was more difficult to compute than just checking graph isomorphism.
- New graph polynomials may appear when we model behavior of physical, chemical or biological systems. The arguments whether a graph polynomial is interesting, depends on its success in predicting the behavior of the modeled systems. Also the particular choice of the representation is dictated by the modeling process. The fact that the modeled process gives, in this case, rise to a particular graph polynomial, is secondary, and the properties of the graph polynomial reflect more properties of the physical or chemical process modeled, than properties of the underlying graph.
- New graph polynomials are also studied as part of graph theory proper. Here one is interested in the interrelationship between various graph parameters without particular applications in mind. A graph polynomial is considered interesting *from a graph theoretic point of view*, if many graph parameters can be (easily) derived from it.
- Graph polynomials are sometimes studied as a way of generating families of polynomials, irrespective of their graph theoretic meaning. H. Wilf, [Wil73] asked the question how to characterize the polynomials which do occur as instances of chromatic polynomials of graphs as a family of polynomials. We have addressed this approach to graph polynomials in [KMR17a].

This paper deals only with the graph theoretic and logical aspects of graph polynomials, discarding the graph isomorphisms problem and discarding the modeling of systems describing phenomena in the natural sciences. We ultimately ask the question: When is a newly introduced graph polynomial interesting from a graph theoretic point of view and deserves to be studied, and what aspects are more rewarding in this study than others. In particular we scrutinize the role of the location of the roots of specific graph polynomials in terms of other graph theoretic properties.

1.3. On the location of roots of graph polynomials

Given a univariate graph polynomial $\mathbf{P}(G; X)$ a complex number $z \in \mathbb{C}$ is a root of \mathbf{P} if there is a graph G such that z is a root of $\mathbf{P}(G; X)$. Many

results in the literature on graph polynomials deal with the location of its roots. For multivariate graph polynomials the corresponding question is formulated in terms of half-plane properties. The last part of this paper shows that the location of roots is not a semantic property.

In this paper we justify, from a foundational point of view, the definitions we have introduced in [MRB14]. This concerns the restrictions of graph polynomials to graph polynomials definable in SOL, the various notions of equivalence of graph polynomials, the notions of syntactic and semantic properties of graph polynomials. We also paraphrase the main results of [MRB14]. These results are all of the form:

(*) Let U be a subset of the complex numbers, such as the reals, an open disk, the lower or upper halfplane, or the complement thereof. Given a univariate SOL-definable graph polynomial $\mathbf{P}(G; X)$, there exists a semantically equivalent SOL-definable graph polynomial $\mathbf{Q}(G; X)$ with all its roots in U .

They show, in a precise sense, that the location of the roots of a univariate graph polynomial is *not a semantic property*. They are more of a *normal form property*: Every univariate SOL-definable graph polynomial $\mathbf{P}(G; X)$ can be put into a semantically equivalent form with prescribed location of its roots.

The proofs in [MRB14] have two parts: Finding $\mathbf{Q}(G; X)$, and showing that this $\mathbf{Q}(G; X)$ is SOL-definable. Finding $\mathbf{Q}(G; X)$ often uses some “dirty trick” from analysis, whereas showing SOL-definability, only sketched in [MRB14], needs more efforts in the details.

In this paper we extend results of [MRB14] to multivariate graph polynomials $\mathbf{P}(G; \mathbf{X})$. We show that various versions of the “halfplane property” in higher dimensions of multivariate graph polynomials are also not semantic properties of the underlying graph in the sense of (*). This is interesting for two reasons: First, these halfplane properties were studied in the recent literature on graph polynomials, and, second, the proofs that the constructed $\mathbf{Q}(G; \mathbf{X})$ is SOL-definable is much more complex. For the convenience of the logically minded reader we repeat many examples already discussed in [MRB14]. Furthermore, we provide in this paper the details in proving SOL-definability for the more difficult case of multivariate graph polynomials and the various halfplane properties.

1.4. Outline of the paper

In Section 2 we discuss the foundational aspects of comparing graph polynomials. In Section 3 we discuss different ways of representing graph polynomials and introduce the notion of semantic (graph theoretic) and syntactic (algebraic) properties of graph polynomials. In Section 4 we present the discussion of various notions of equivalence of graph polynomials based on their distinctive power which also is part of [KMR17a]. In Section 5 we develop the framework of SOL-definable graph polynomials. This summarizes the framework given in [Kot12]. In Section 6 we discuss the location of zeros of graph polynomials.

First, in Subsection 6.1, we review our previous results previously published in [MRB14], which show that the location of roots of univariate graph polynomials is not a semantic property. Then, in Subsection 6.2, we look at the multivariate version of the location of roots, the various halfplane properties, also called stability properties, and prove that stability is also not a semantic property of multivariate graph polynomials. The discussion of stable polynomials is appears for the first time in this paper. Finally, in Section 7 we draw our conclusions and formulate several open problems.

2. How to compare graph polynomials?

Once the graph theorists started to study several graph polynomials, the need of comparing them naturally arises.

For $\mathcal{R} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Z}\}$ we denote by $\mathfrak{GP}_{\mathcal{R}, r}$ the set of graph polynomials in r indeterminates with coefficients in \mathcal{R} , and let $\mathbf{X} = (X_1, \dots, X_r)$ be r indeterminates. Let $\mathbf{P}(G) = \mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G) = \mathbf{Q}(G; \mathbf{X})$ be two graph polynomials.

The following statements appear frequently in the literature with the *intended meaning*, but *without a general definition*:

- (i) $\mathbf{Q}(G)$ is a *substitution instance* of $\mathbf{P}(G)$.
- (ii) Q and P are really the same, up to a *prefactor*. For example the various versions of the Tutte polynomial are said to be the same *up to a prefactor*, [Sok05], and the same holds for the various versions of the matching polynomial, [LP86].
- (iii) Q is *at least as expressive* than P .
- (iv) The coefficients of $\mathbf{P}(G)$ can be determined, or even computed, from the coefficients of $\mathbf{Q}(G)$.

Usually these statements are understood to be uniform in the graphs G , but this uniformity can take various forms. In [MRB14] we have given these statements precise meanings, and we have initiated the analysis of their relationship. In this paper we elaborate our approach from [MRB14] further with the logic community in mind.

From a model theoretic point of view a graph property is a Boolean graph parameter. A closed formula in a logical formalism, say a fragment of SOL, is a syntactic object. Its meaning is given by a graph property, i.e., a class of finite graphs closed under isomorphism. Two formulas are considered *logically equivalent* if they define the same property. In other words, two formulas are considered equivalent if they do not distinguish between two graphs. As Boolean graph parameters have only two possible values, two formulas are equivalent if, considered as graph parameters, they define the same function.

Let \mathcal{R} be a possibly infinite ring. An \mathcal{R} -valued graph parameter is a function f which maps a graph G into an element $f(G) \in \mathcal{R}$. A graph polynomial \mathbf{P} is a graph parameter which takes values in a polynomial ring.

Graph parameters are *coextensive* if they define the same function. However, coextensiveness seems to be too strong a property to compare graph parameters.

For instance defining the size of a graph G by its order $n(G) = |V(G)|$, or by $n'(G) = 2 \cdot |V(G)|$, gives two non-coextensive graph parameters which still have the same information content in the following sense. For two \mathcal{R} -valued graph parameters f and g , we say that g is at least as distinctive as f , if for two graphs G_1, G_2 g does not distinguish between G_1 and G_2 , i.e., $g(G_1) = g(G_2)$, then also f does not distinguish between G_1 and G_2 , i.e., $f(G_1) = f(G_2)$.

Graph theorists often compare the distinctive power of graph parameters on graphs which are not trivially distinguishable. Here trivially distinguishable refers to different order, size or number of components.

2.1. Equivalence of graph polynomials

Let \mathbf{P} be a graph polynomial. We say that two graphs G, H are *similar* if they have the same number of vertices, edges and connected components. A graph parameter or a graph polynomial is a *similarity function* if it is invariant under graph similarity.

Two graphs G, H are \mathbf{P} -equivalent if $\mathbf{P}(H; \mathbf{X}) = \mathbf{P}(G; \mathbf{X})$. \mathbf{P} distinguishes between G and H if G and H are not \mathbf{P} -equivalent. Two graph polynomials $\mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G; \mathbf{Y})$ with r and s indeterminates respectively can be compared by their *distinctive power* on similar graphs: \mathbf{P} is at most as distinctive as \mathbf{Q} , $\mathbf{P} \leq_{s.d.p.} \mathbf{Q}$ if any two similar graphs G, H which are \mathbf{Q} -equivalent are also \mathbf{P} -equivalent. \mathbf{P} and \mathbf{Q} are *s.d.p.-equivalent*, $\mathbf{P} \sim_{s.d.p.} \mathbf{Q}$ if for any two similar graphs G, H \mathbf{P} -equivalence and \mathbf{Q} -equivalence coincide. We can also compare graph polynomials on graphs without requiring similarity. In this case we say that a graph polynomial \mathbf{P} is at most as distinctive as \mathbf{Q} , $\mathbf{P} \leq_{d.p.} \mathbf{Q}$, if for all graphs G_1 and G_2 we have that

$$\mathbf{Q}(G_1) = \mathbf{Q}(G_2) \text{ implies } \mathbf{P}(G_1) = \mathbf{P}(G_2)$$

\mathbf{P} and \mathbf{Q} are d.p.-equivalent iff both $\mathbf{P} \leq_{d.p.} \mathbf{Q}$ and $\mathbf{Q} \leq_{d.p.} \mathbf{P}$. D.p.-equivalence is stronger than s.d.p.-equivalence:

Lemma 2.1. *For any two graph polynomials \mathbf{P} and \mathbf{Q} we have: $\mathbf{P} \leq_{d.p.} \mathbf{Q}$ implies $\mathbf{P} \leq_{s.d.p.} \mathbf{Q}$.*

A graph G is \mathbf{P} -unique if for all graphs G' the polynomial identity $\mathbf{P}(G; \mathbf{X}) = \mathbf{P}(G'; \mathbf{X})$ implies that G is isomorphic to G' . As a graph invariant $\mathbf{P}(G; \mathbf{X})$ can be used to check whether two graphs are not isomorphic. For \mathbf{P} -unique graphs G and G' the polynomial $\mathbf{P}(G; \mathbf{X})$ can also be used to check whether they are isomorphic.

Our notion of similarity is extracted from the literature on graph polynomials: It is implicitly used frequently both in claims that two polynomials are “really the same”, or “the same up to a prefactor”. From a logical point of view one would rather define a more general notion: Let Σ be a finite set of graph parameters. Two graphs G, H are Σ -similar if they have the same values $s(G) = s(H)$ for all $s \in \Sigma$. It is easy, but currently of little use, to rewrite the definitions of various forms of equivalence of graph polynomials using Σ -similarity rather than similarity as we defined it in this paper.

Theorem 2.2. (i) \mathbf{P} is at most as distinctive as \mathbf{Q} , $\mathbf{P} \leq_{d.p.} \mathbf{Q}$, iff there is a function $F : \mathbb{Z}[\mathbf{Y}] \rightarrow \mathbb{Z}[\mathbf{X}]$ such that for every graph G we have

$$\mathbf{P}(G; \mathbf{X}) = F(\mathbf{Q}(G; \mathbf{Y}))$$

(ii) \mathbf{P} is at most as distinctive as \mathbf{Q} on similar graphs, $\mathbf{P} \leq_{s.d.p.} \mathbf{Q}$, iff there is a function $F : \mathbb{Z}[\mathbf{Y}] \times \mathbb{Z}^3 \rightarrow \mathbb{Z}[\mathbf{X}]$ such that for every graph G we have

$$\mathbf{P}(G; \mathbf{X}) = F(\mathbf{Q}(G; \mathbf{Y}), n(G), m(G), k(G))$$

(iii) Furthermore, both for d.p. and s.d.p., if both \mathbf{P} and \mathbf{Q} are computable, then F is computable, too.

The equivalence in (ii) in Theorem 2.2 was first proved in [MRB14]. For the convenience of the reader we repeat it below. Moreover, (iii) is new, and follows from the our definition of computability of graph polynomials. We note here that (ii) is useful for proving d.p.-reducibility, whereas (i) is more useful to prove its negation. Theorem 2.2 shows that our definition of d.p.-equivalence of graph polynomials is mathematically equivalent to the definition proposed in [MN09].

Proof of Theorem 2.2(ii)-(iii).

(ii) \Rightarrow :

Let S be a set of finite graphs and $s \in \mathbb{Z}[\mathbf{X}]$. For a graph polynomial \mathbf{P} we define:

$$\begin{aligned} \mathbf{P}[S] &= \{s \in \mathbb{Z}[\mathbf{X}] : \mathbf{P}(G) = s \text{ for some } G \in S\} \\ \mathbf{P}^{-1}(s) &= \{G : \mathbf{P}(G) = s\}. \end{aligned}$$

Now assume $\mathbf{P}(G; \mathbf{X}) \preceq_{s.d.p.} \mathbf{Q}(G; \mathbf{Y})$.

If $Q^{-1}(s) \neq \emptyset$, then for every $G_1, G_2 \in Q^{-1}(s)$ we have $\mathbf{Q}(G_1) = \mathbf{Q}(G_2)$, and therefore $\mathbf{P}(G_1) = \mathbf{P}(G_2)$. Hence $P[Q^{-1}(s)] = \{t_s\}$ for some $t_s \in \mathbb{Z}[\mathbf{X}]$. Now we define

$$F_{P,Q}(s) = \begin{cases} t_s & Q^{-1}(s) \neq \emptyset \\ s & \text{else} \end{cases}$$

\Leftarrow :

Assume there is a function $F : \mathbb{Z}[\mathbf{Y}] \rightarrow \mathbb{Z}[\mathbf{X}]$ such that for all graphs G we have $F(\mathbf{Q}(G)) = \mathbf{P}(G)$.

Now let G_1, G_2 be similar graphs such that $\mathbf{Q}(G_1) = \mathbf{Q}(G_2)$. Hence, $F(\mathbf{Q}(G_1)) = F(\mathbf{Q}(G_2))$. Since for all G we have $F(\mathbf{Q}(G)) = \mathbf{P}(G)$, we get $\mathbf{P}(G_1) = \mathbf{P}(G_2)$.

Proof of (iii): Now assume both $\mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G; \mathbf{Y})$ are computable. Therefore, both $P : \mathcal{G} \rightarrow \mathbb{Z}[\mathbf{X}]$ and $\beta_Q : \mathbb{Z}[\mathbf{Y}] \rightarrow \mathbb{N}$ are computable. To see that F is computable we note that it suffices, as in the proof of (i), to find an element in the range of Q and a graph suitable for it. The latter can be done in exponential time in the size of $\beta_Q(s)$. \square

2.2. Examples of equivalent graph polynomials

Example 2.3. Let $m_k(G)$ denote the number of k -matchings (k many independent edges) of G . There are two versions of the univariate matching polynomial, [LP86]: The matching defect polynomial (or acyclic polynomial)

$$\mu(G; X) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(G) X^{n-2k},$$

and the matching generating polynomial

$$g(G; X) = \sum_{k=0}^n m_k(G) X^k.$$

The relationship between the two is given by

$$\begin{aligned} \mu(G; X) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(G) X^{n-2k} = X^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(G) X^{-2k} = \\ &= X^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(G) ((-1) \cdot X^{-2})^k = X^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(G) (-X^{-2})^k = X^n g(G; (-X^{-2})) \end{aligned}$$

It follows that g and μ are equally distinctive, and even mutually prefactor reducible. However, $g(G; X)$ is invariant under addition or removal of isolated vertices, whereas $\mu(G; X)$ counts them.

Example 2.4. Let $G = (V(G), E(G))$ be a loopless graph without multiple edges. Let A_G be the adjacency matrix of G , D_G the diagonal matrix with $(D_G)_{i,i} = d(i)$, the degree of the vertex i , and $L_G = D_G - A_G$. In spectral graph theory two graph polynomials are considered, the characteristic polynomial of G , here denoted by $P_A(G; X) = \det(X \cdot \mathbb{I} - A_G)$, and the Laplacian polynomial, here denoted by $P_L(G; X) = \det(X \cdot \mathbb{I} - L_G)$. Here \mathbb{I} denotes the unit element in the corresponding matrix ring. Here we show that the polynomials $P_{cc}(G; X)$ and $P_L(G; X)$ are d.p.-incomparable. G and H in Figure 1 are similar. We have

$$P_{cc}(G; X) = P_{cc}(H; X) = (X - 1)(X + 1)^2(X^3 - X^2 - 5X + 1),$$

but G has eight spanning trees, and H has six. Therefore, $P_L(G; X) \neq P_L(H; X)$, as one can compute the number of spanning trees from $P_L(G; X)$. For more details, cf. [BH12, Exercise 1.9].

On the other hand, G' and H' in Figure 2 are similar, but G' is not bipartite, whereas, H' is. Hence $P_A(H'; X) \neq P_A(G'; X)$, but $P_L(H'; X) = P_L(G'; X)$. See, [BH12, Lemma 14.4.3].

Conclusion: The characteristic polynomial and the Laplacian polynomial are d.p.-incomparable. However, if restricted to r -regular graphs, they are d.p.-equivalent, [BH12].



Figure 1:



Figure 2:

2.3. Prefactor equivalence

We recall that a graph parameter $f(G)$ with values in some function space \mathbf{F} over some ring \mathcal{R} is called a *similarity function* if for any two similar graphs G, H we have that $f(G) = f(H)$. If \mathbf{F} is a subset of the set of analytic functions we speak of *analytic similarity functions*.

If \mathbf{F} is the polynomial ring $\mathbb{Z}[\mathbf{X}]$ with set of indeterminates $\mathbf{X} = (X_1, \dots, X_r)$, we speak of *similarity polynomials*. It will be sometimes useful to allow classes of functions spaces which are closed under *reciprocals* and *inverses* rather than just similarity polynomials.

Example 2.5. *Typical examples of similarity functions are*

- (i) *The nullity $\nu(G) = m(G) - n(G) + k(G)$ and the rank $\rho(G) = n(G) - k(G)$ of a graph G are similarity polynomials with integer coefficients.*
- (ii) *Similarity polynomials can be formed inductively starting with similarity functions $f(G)$ not involving indeterminates, and monomials of the form $X^{g(G)}$, where X is an indeterminate and $g(G)$ is a similarity function not involving indeterminates. One then closes under pointwise addition, subtraction, multiplication and substitution of indeterminates X by similarity polynomials.*
- (iii) *$f(G; X) = n(G)X^2$ is a similarity polynomial with integer coefficients. Its inverse $f^{-1}(G; X) = n(G)^{-1}X^{\frac{1}{2}}$ is analytic at any point $a \in \mathbb{R}$ with $a \neq 0$. Its reciprocal $\frac{1}{f(G; X)}$ is rational.*

In the literature one often wants to say that two graph polynomials are *almost the same*. We propose a definition which makes this precise.

Definition 2.6. *Let $\mathbf{P}(G; Y_1, \dots, Y_r)$ and $\mathbf{Q}(G; X_1, \dots, X_s)$ be two multivariate graph polynomials with coefficients in a ring \mathcal{R} .*

- (i) *We say that $\mathbf{P}(G; \mathbf{Y})$ is prefactor reducible to $\mathbf{Q}(G; \mathbf{X})$ over a set of similarity functions \mathbf{F} , and we write*

$$\mathbf{P}(G; \mathbf{Y}) \preceq_{\text{prefactor}}^{\mathbf{F}} \mathbf{Q}(G; \mathbf{X})$$

if there are similarity functions $f(G; \mathbf{X})$ and $g_i(G; \mathbf{X}), i \leq r$ in \mathbf{F} such that

$$\mathbf{P}(G; \mathbf{Y}) = f(G; \mathbf{X}) \cdot \mathbf{Q}(G; g_1(G; \mathbf{Y}), \dots, g_r(G; \mathbf{Y}))$$

- (ii) We say that $\mathbf{P}(G; \mathbf{Y})$ is substitution reducible to $\mathbf{Q}(G; \mathbf{X})$ over \mathbf{F} and we write

$$\mathbf{P}(G; \mathbf{Y}) \preceq_{\text{subst}} \mathbf{Q}(G; \mathbf{X})$$

if $f(G; \mathbf{X}) = 1$ is the constant function for all graphs G .

- (iii) We say that $\mathbf{P}(G; \mathbf{Y})$ and $\mathbf{Q}(G; \mathbf{X})$ are prefactor equivalent, and we write

$$\mathbf{P}(G; \mathbf{Y}) \sim_{\text{prefactor}} \mathbf{Q}(G; \mathbf{X})$$

if the relation holds in both directions.

- (iv) substitution equivalence $\mathbf{P}(G; \mathbf{Y}) \sim_{\text{subst}} \mathbf{Q}(G; \mathbf{X})$ is defined analogously.

The following properties follow from the definitions.

Proposition 2.7. Assume we have two graph polynomials $\mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G; \mathbf{X})$. For reducibilities we have:

- (i) $\mathbf{P}(G; \mathbf{X}) \preceq_{\text{subst}} \mathbf{Q}(G; \mathbf{X})$ implies $\mathbf{P}(G; \mathbf{X}) \preceq_{\text{prefactor}} \mathbf{Q}(G; \mathbf{X})$.
- (ii) $\mathbf{P}(G; \mathbf{X}) \preceq_{\text{prefactor}} \mathbf{Q}(G; \mathbf{X})$ implies $\mathbf{P}(G; \mathbf{X}) \preceq_{\text{s.d.p.}} \mathbf{Q}(G; \mathbf{X})$.

The corresponding implications for equivalence obviously also hold.

2.4. The classical examples

Example 2.8 (The universal Tutte polynomial). Let $T(G; X, Y)$ be the Tutte polynomial, [Bol98, Chapter 10]. The universal Tutte polynomial is defined by

$$U(G; X, Y, U, V, W) = U^{k(G)} \cdot V^{\nu(G)} \cdot W^{\rho(G)} \cdot T\left(G; \frac{UX}{W}, \frac{Y}{U}\right).$$

$U(G; X, Y, U, V, W)$ is the most general graph polynomial satisfying the recurrence relations of the Tutte polynomial in the sense that every other graph polynomial satisfying these recurrence relations is a substitution instance of $U(G; X, Y, U, V, W)$.

Here, $\nu(G) = m(G) - n(G) + k(G)$ is the nullity of G , and $\rho(G) = n(G) - k(G)$ is the rank of G . Clearly, $U(G; X, Y, U, V, W)$ is prefactor equivalent to $T(G; X, Y)$ using rational similarity functions.

Example 2.9 (The matching polynomials). In Example 2.3 we have already seen the three matching polynomials:

$$\mu(G; X) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i m_i(G) X^{n(G)-2i}$$

$$g(G; Y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} m_i(G) Y^i$$

$$M(G; X, Y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} m_i(G) X^i Y^{n(G)-2i}$$

We have $\mu(G; X) = X^{n(G)} \cdot g(G; -X^{-2})$ and $M(G; X, Y) = Y^{n(G)} \cdot g(G; \frac{X}{Y^2})$. Clearly, all three matching polynomials are mutually prefactor bi-reducible using analytic similarity functions.

Example 2.10. *The following graph polynomials are d.p.-equivalent but incomparable by prefactor reducibility:*

- (i) $M(G; X)$ and $M(G; X)^2$;
- (ii) $\mu(G; X)$ and $\sum_i m_i(G) \binom{X}{i}$.

In the literature there are at least two theorems which state that two graph polynomials have the same coefficients if restricted to some graph class \mathcal{K} .

Theorem 2.11 (C.D. Godsil, I. Gutman, [GG81]). *Let $\mu(G; X)$ be the defect matching polynomial and $P_{cc}(G; X)$ the characteristic polynomial. Let \mathcal{F} be the class of forests. Then for every graph in \mathcal{F} we have that $\mu(G; X)$ is d.p.-equivalent to $P_{cc}(G; X)$ and even stronger, that*

$$\mu(G; X) = P_{cc}(G; X).$$

Let

$$M(G; X, Y) = \sum_i m_i(G) X^i Y^{n(G)-2i}$$

be the bivariate matching polynomial, and

$$M'(G; X) = \sum_i m_i(G) X^i X^{n(G)-2i} = \sum_i m_i(G) X^{n(G)-i}$$

its substitution instance for $Y = X$. Furthermore let

$$\hat{\chi}(G; X) = \chi(\bar{G}; X)$$

be the chromatic polynomial of the complement graph of G .

Remark 2.12. (i) $M(G; X, Y)$ is d.p.-equivalent to $M'(G; X)$ using a simple substitution.

- (ii) $\chi(G; X)$ and $\hat{\chi}(G; X)$ are d.p.-incomparable. To see this, we note that for any number $m \in \mathbb{N}$ the polynomial $\chi(G; X)$ evaluated at $X = m$ does not distinguish cliques of size bigger than m , whereas $\hat{\chi}(G; X)$ evaluated at $X = m$ does distinguish between them.

Theorem 2.13 (E.J. Farrell, E.G. Whitehead, [FW92]). *Let Δ be the class of triangle-free graphs. Then for each $G \in \Delta$ we have that $M'(G; X)$ is d.p.-equivalent to $\hat{\chi}(G; X)$ and even stronger, that*

$$M'(G; X) = \hat{\chi}(G; X).$$

In both theorems the equality of the polynomials is semantically meaningless.

3. How to represent graph polynomials?

3.1. Choosing a basis in the polynomial ring

Example 3.1. Let $d(G)$ be a graph parameter, and let $\mathbf{P}(G; X)$ be a univariate graph polynomial with integer coefficients.

(i) Assume

$$\begin{aligned}\mathbf{P}(G; X) &= \sum_{i=0}^{d(G)} a_i(G) X^i = \sum_{i=0}^{d(G)} b_i(G) X_{(i)} = \\ &= \sum_{i=0}^{d(G)} c_i(G) X^{(i)} = \sum_{i=0}^{d(G)} e_i(G) \binom{X}{i} = \prod_i^{d(G)} (X - z_i(G))\end{aligned}$$

where

$$X_{(i)} = X(X-1) \cdot \dots \cdot (X-i+1)$$

is the falling factorial function, and

$$X^{(i)} = X(X+1) \cdot \dots \cdot (X+i)$$

is the rising factorial function. and z_i are its roots. Clearly, these are different presentations of the same polynomial, hence they are all d.p.-equivalent.

(ii) Now look at the polynomials below, where the coefficients remain the same, but the polynomial basis is changed:

$$\mathbf{P}(G; X) = P_0(G; X) = \sum_{i=0}^{d(G)} a_i(G) X^i = \prod_i^{d(G)} (X - z_i(G)) \quad (1)$$

$$P_1(G; X) = \sum_{i=0}^{d(G)} a_i(G) X_{(i)} \quad (2)$$

$$P_2(G; X) = \sum_{i=0}^{d(G)} a_i(G) X^{(i)} \quad (3)$$

$$P_3(G; X) = \sum_{i=0}^{d(G)} a_i(G) \binom{X}{i} \quad (4)$$

Obviously, $P_i(G; X)$ are different polynomials which have different roots, but by Theorem 2.2 they are all d.p.-equivalent.

Example 3.1 shows that the location of the roots of a graph polynomial is not invariant under d.p.-equivalence.

The notion of d.p.-equivalence (having the same distinguishing power) of graph polynomials evolved very slowly, mostly in implicit arguments. Originally, a graph polynomial such as the chromatic or characteristic polynomial had

a *unique* definition which both determined its algebraic presentation and its semantic content. The need to spell out semantic equivalence emerged when the various forms of the Tutte polynomial had to be compared. As it was to be expected, some of the presentations of the Tutte polynomial had more convenient properties than others, and some of the properties of one form got completely lost when passing to another semantically equivalent form.

Two d.p.-equivalent polynomials carry the same combinatorial information about the underlying graph, independently of their presentation as polynomials. This situation is analogous to the situation in Linear Algebra: Similar matrices represent the same linear operator under two different bases. The choice of a suitable basis, however, may be useful for numeric evaluations. Here d.p.-equivalent graph polynomials represent the same combinatorial information under two different polynomial representations. The choice of a particular polynomial representation $\mathbf{P}(G; \mathbf{X})$ may carry more numeric information about a particular graph parameter $p(G)$ determined by $\mathbf{P}(G; \mathbf{X})$.

3.2. Typical forms of graph polynomials

In this paper we look at five types of graph polynomials: generalized chromatic polynomials and polynomials defined as generating functions of induced or spanning subgraphs, and determinant polynomials, and contrast this to graph polynomials arising from generating functions of relations.

More precisely, let \mathcal{C} be a graph property.

Generalized chromatic: Let $\chi_{\mathcal{C}}(G; k)$ denote the number of colorings of G with at most k colors such that each color class induces a graph in \mathcal{C} . It was shown in [KMZ08, KMZ11] that $\chi_{\mathcal{C}}(G; k)$ is a polynomial in k for any graph property \mathcal{C} . Generalized chromatic polynomials are further studied in [GHK⁺17].

Generating functions: Let $A \subseteq V(G)$ and $B \subseteq E(G)$. We denote by $G[A]$ the induced subgraph of G with vertices in A , and by $G\langle B \rangle$ the spanning subgraph of G with edges in B .

(i) Let \mathcal{C} be a graph property.

$$P_{\mathcal{C}}^{ind}(G; X) = \sum_{A \subseteq V: G[A] \in \mathcal{C}} X^{|A|}$$

(ii) Let \mathcal{D} be a graph property which is closed under adding isolated vertices, i.e., if $G \in \mathcal{D}$ then $G \sqcup K_1 \in \mathcal{D}$.

$$P_{\mathcal{D}}^{span}(G; X) = \sum_{B \subseteq E: G\langle B \rangle \in \mathcal{D}} X^{|B|}$$

Generalized Generating functions: Let $X_i : i \leq r$ be indeterminates and $f_i : i \leq r$ be graph parameters. We also consider graph polynomials of the form

$$P_{\mathcal{C}, f_1, \dots, f_r}^{ind}(G; X) = \sum_{A \subseteq V: G[A] \in \mathcal{C}} \prod_{i=1}^r X_i^{f_i(G[A])}$$

and

$$P_{\mathcal{C}, f_1, \dots, f_r}^{span}(G; X) = \sum_{B \subseteq E: G \langle B \rangle \in \mathcal{D}} \prod_{i=1}^r X_i^{f_i(G \langle B \rangle)}$$

Determinants: Let M_G be a matrix associated with a graph G , such as the adjacency matrix, the Laplacian, etc. Then we can form the polynomial $\det(\mathbf{1} \cdot X - M_G)$.

Special cases are the chromatic polynomial $\chi(G; X)$, the independence polynomial $I(G; X)$, the Tutte polynomial $T(G; X, Y)$ and the characteristic polynomial of a graph $p_{char}(G; X)$. Note that, in the sense of the following subsection, $\chi(G; X)$, $I(G; X)$ and $p_{char}(G; X)$ are mutually d.p.-incomparable, and $\chi(G; X)$ has strictly less distinctive power than $T(G; X, Y)$.

In Section 4.6 we shall see that there are graph polynomials defined in the literature which seemingly do not fit the above frameworks. This is the case for the usual definition of the *generating matching polynomial*:

$$\sum_{M \subseteq E(G): \text{match}(M)} X^{|M|}$$

where $\text{match}(M)$ says that $(V(G), M)$ is a matching, i.e., M is a set of isolated edges in G . However, we shall see in Section 4.6 that there is another definition of the same polynomial which is an generating function. In stark contrast to this, we shall prove there, that the dominating polynomial

$$DOM(G; X) = \sum_{A \subseteq V(G): \Phi_{dom}(A)} X^{|A|}$$

where $\Phi_{dom}(A)$ says that A is a dominating set of G , cannot be written as a generating function, (Theorem 4.18). This motivates the next definition, see also Section 4.6.

Generating functions of a relation Let Φ be a property of pairs (G, A) where G is a graph and $A \subseteq V(G)^r$ is an r -ary relation on G . Then the generating function of Φ is defined by

$$\mathbf{P}_\Phi(G; X) = \sum_{A \subseteq V(G)^r: \Phi(G, A)} X^{|A|}$$

The most general graph polynomials Further generalizations of chromatic polynomials were studied in [MZ06, KMZ11, Kot12] and in [GGN13, GNdM16a]. In [MZ06, KMZ11] it was shown that the most general graph polynomials can be obtained using model theory as developed in [Zil93, CH03]. A similar approach was used in [GGN13, GNdM16b] based on ideas from [dlHJ95]. However, for our presentation here, the graph polynomials we have defined so far suffice.

3.3. Syntactic vs semantic properties of graph polynomials

An n -ary property of graph polynomials Φ , aka a *GP-property*, is a subset of $\mathfrak{GP}_{\mathcal{R},m}^n$. Φ is a *semantic property* if it is closed under d.p.-equivalence. Semantic properties are independent of the particular presentation of its members. Consequently, we call a property Φ , which does depend on the presentation of its members, a *syntactic (aka algebraic) property*. Let us make this definition clearer via examples:

Examples 3.2. (i) *The GP-property which says that for every graph G the polynomial $\mathbf{P}(G, \mathbf{X})$ is P -unique, is a semantic property.*

(ii) *The unary GP-properties of univariate graph polynomials that for each graph G the polynomials $\mathbf{P}(G; X)$ is monic⁵, or that its coefficients are unimodal⁶, is not a semantic GP-property, because, by applying Theorem 2.2, multiplying each coefficient by a fixed integer gives a d.p.-equivalent graph polynomial.*

(iii) *The GP-property that the multiplicity of a certain value a as a root of $\mathbf{P}(G; X)$ coincides with the value of a graph parameter $p(G)$ with values in \mathbb{N} , is not a semantic property. For example, the multiplicity of 0 as a root of the Laplacian polynomial is the number of connected components $k(G)$ of G , [BH12, Chapter 1.3.7]. However, stating that for two graphs G_1, G_2 with $\mathbf{P}(G_1; X) = \mathbf{P}(G_2; X)$ we also have $p(G_1) = p(G_2)$, is a semantic property.*

(iv) *Similarly, proving that the leading coefficient of $\mathbf{P}(G; \mathbf{X})$ equals the number of vertices of G is not a semantic property, for the same reason. However, proving that two graphs G_1, G_2 with $\mathbf{P}(G_1; \mathbf{X}) = \mathbf{P}(G_2; \mathbf{X})$ have the same number of vertices is semantically meaningful.*

(v) *In similar vain, the classical result of [GG81], that the characteristic polynomial of a forest equals the (acyclic) matching polynomial of the same forest, is a syntactic coincidence, or reflects a clever choice in the definition of the acyclic matching polynomial, but it is not a semantic GP-property. The semantic GP-property of this result says that if we restrict our graphs to forests, then the characteristic and the matching polynomials (in all its versions) have the same distinctive power on trees of the same size. We discuss this and similar examples further in [MRB14].*

To prove a semantic GP-property it is sometimes easier to prove a stronger non-semantic version. From the above examples, (iii), (iv) and (v) are illustrative cases for this.

To motivate our definition of d.p.-equivalence we first give the examples taken from [MRB14]. For the multivariate case we use [ATME11] which shows that the universal EE-polynomial $\xi(G, X, Y, Z)$ and the component polynomial

⁵A univariate polynomial is monic if the leading coefficient equals 1.

⁶A sequence of numbers $a_i : i \leq m$ is unimodal if there is $k \leq m$ such that $a_i \leq a_j$ for $i < j < k$ and $a_i \geq a_j$ for $k \leq i < j \leq m$.

$C(G; X, Y, Z)$ from [Tri12] are d.p.-equivalent and are comparable and more expressive than the Tutte polynomial, the matching polynomials, the independent set polynomial, and the chromatic polynomial.

4. Distinctive power

4.1. s.d.p.-equivalence and d.p.-equivalence of graph properties

We recall that a class of graphs \mathcal{S} which consists of all graphs having the same number of vertices, edges and connected components is called a *similarity class*.

Let \mathcal{C} be a graph property. Two graphs G, H are \mathcal{C} -equivalent if either both are in \mathcal{C} or both are not in \mathcal{C} . We denote by $\bar{\mathcal{C}}$ the graph property $\mathcal{G} - \mathcal{C}$.

Therefore we have:

- Proposition 4.1.** (i) *Two graph properties \mathcal{C}_1 and \mathcal{C}_2 are d.p.-equivalent iff either $\mathcal{C}_1 = \mathcal{C}_2$ or $\mathcal{C}_1 = \bar{\mathcal{C}}_2$.*
(ii) *Two graph properties \mathcal{C}_1 and \mathcal{C}_2 are s.d.p.-equivalent iff for every similarity class \mathcal{S} either $\mathcal{C}_1 \cap \mathcal{S} = \mathcal{C}_2 \cap \mathcal{S}$ or $\mathcal{C}_1 \cap \mathcal{S} = \bar{\mathcal{C}}_2 \cap \mathcal{S}$.*

Proof. It is straightforward that if \mathcal{C}_1 and \mathcal{C}_2 are d.p.-equivalent then $\mathcal{C}_2 \cap \mathcal{S} = \mathcal{C}_1 \cap \mathcal{S}$ or $\mathcal{C}_2 \cap \mathcal{S} = \bar{\mathcal{C}}_1 \cap \mathcal{S}$.

For the other direction, we prove first that $\mathcal{C}_1 \cap \mathcal{S} \subseteq \mathcal{C}_2 \cap \mathcal{S}$ or $\mathcal{C}_1 \cap \mathcal{S} \subseteq \bar{\mathcal{C}}_2 \cap \mathcal{S}$. By a symmetrical argument, we then prove also $\mathcal{C}_2 \subseteq \mathcal{C}_1$ or $\mathcal{C}_2 \subseteq \bar{\mathcal{C}}_1$, $\bar{\mathcal{C}}_1 \subseteq \mathcal{C}_2$ or $\bar{\mathcal{C}}_1 \subseteq \bar{\mathcal{C}}_2$ and $\bar{\mathcal{C}}_2 \subseteq \mathcal{C}_1$ or $\bar{\mathcal{C}}_2 \subseteq \bar{\mathcal{C}}_1$. Now the result follows. \square

Remark 4.2. *If \mathcal{C}_1 and \mathcal{C}_2 are s.d.p.-equivalent it is possible that for a similarity class \mathcal{S} we have $\mathcal{C}_1 = \mathcal{C}_2$ but for another similarity class \mathcal{S}' we have $\mathcal{C}_1 = \bar{\mathcal{C}}_2$.*

- Proposition 4.3.** (i) *Let \mathcal{C}_1 and \mathcal{C}_2 be two graph properties. Assume that both \mathcal{C}_1 and \mathcal{C}_1 are not empty and do not contain all finite graphs, and that $\mathcal{C}_1 \neq \mathcal{C}_2$ and $\mathcal{C}_1 \neq \bar{\mathcal{C}}_2$. Then \mathcal{C}_1 and \mathcal{C}_2 are s.d.p.-incomparable, i.e., $\mathcal{C}_1 \not\leq_{d.p.} \mathcal{C}_2$ and $\mathcal{C}_2 \not\leq_{d.p.} \mathcal{C}_1$.*
(ii) *Let \mathcal{C}_1 and \mathcal{C}_2 be two graph properties. Assume there is a similarity class \mathcal{S} such that both $\mathcal{C}_1 \cap \mathcal{S}$ and $\mathcal{C}_1 \cap \mathcal{S}$ are not empty and do not contain all finite graphs in \mathcal{S} , and that $\mathcal{C}_1 \cap \mathcal{S} \neq \mathcal{C}_2 \cap \mathcal{S}$ and $\mathcal{C}_1 \cap \mathcal{S} \neq \bar{\mathcal{C}}_2 \cap \mathcal{S}$. Then \mathcal{C}_1 and \mathcal{C}_2 are s.d.p.-incomparable, i.e., $\mathcal{C}_1 \not\leq_{s.d.p.} \mathcal{C}_2$ and $\mathcal{C}_2 \not\leq_{s.d.p.} \mathcal{C}_1$.*

Proof. We prove only (i) and leave the proof of (ii) to the reader. Assume $G_1 \in (\mathcal{C}_1 - \mathcal{C}_2) \cap \mathcal{S}$, $G_2 \in (\mathcal{C}_2 - \mathcal{C}_1) \cap \mathcal{S}$ and $G_3 \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{S}$, the other cases being similar. Then $G_2, G_3 \in \mathcal{C}_2 \cap \mathcal{S}$. If $\mathcal{C}_1 \leq_{d.p.}$, we would have that both $G_2, G_3 \in \mathcal{C}_1 \cap \mathcal{S}$, or both $G_2, G_3 \notin \mathcal{C}_1 \cap \mathcal{S}$, a contradiction. \square

In the next two subsections we look at graph polynomials, which are either generating functions, or count colorings which, in both cases, solely depend on a graph property \mathcal{C} .

4.2. Graph polynomials as generating functions

Let \mathcal{C} be a graph property, and \mathcal{D} be a graph property closed under adding and removal isolated vertices. Recall from Section 3.2 the definitions

$$\mathbf{P}_{\mathcal{C}}^{ind}(G; X) = \sum_{A \subseteq V: G[A] \in \mathcal{C}} X^{|A|} \quad \text{and} \quad \mathbf{P}_{\mathcal{D}}^{span}(G; X) = \sum_{B \subseteq E: G\langle B \rangle \in \mathcal{D}} X^{|B|}.$$

Let $|V(G)| = n(G)$ and $|E(G)| = m(G)$.

Proposition 4.4. (i) $\mathcal{C} \leq_{d.p.} \mathbf{P}_{\mathcal{C}}^{ind}(G; X)$ and
(ii) $\mathcal{D} \leq_{d.p.} \mathbf{P}_{\mathcal{D}}^{span}(G; X)$.

Proof. (i) follows from the fact that $G \in \mathcal{C}$ iff the coefficient of $X^{n(G)}$ in $\mathbf{P}_{\mathcal{C}}^{ind}(G; X)$ does not vanish.

Similarly, (ii) follows from the fact that $G \in \mathcal{D}$ iff the coefficient of $X^{m(G)}$ in $\mathbf{P}_{\mathcal{D}}^{span}(G; X)$ does not vanish. \square

From Lemma 2.1 we get immediately:

Corollary 4.5. (i) $\mathcal{C} \leq_{s.d.p.} \mathbf{P}_{\mathcal{C}}^{ind}(G; X)$ and
(ii) $\mathcal{D} \leq_{s.d.p.} \mathbf{P}_{\mathcal{D}}^{span}(G; X)$.

Proposition 4.6. With $|V(G)| = n(G)$ and $|E(G)| = m(G)$ we have:

- (i) $\mathbf{P}_{\mathcal{C}}^{ind}(G; X) + \mathbf{P}_{\mathcal{C}}^{ind}(G; X) = (1 + X)^{n(G)}$
- (ii) $\mathbf{P}_{\mathcal{D}}^{span}(G; X) + \mathbf{P}_{\mathcal{D}}^{span}(G; X) = (1 + X)^{m(G)}$

Proof. (i): Put

$$c_i(G) = |\{A \subseteq V(G) : |A| = i, G[A] \in \mathcal{C}\}|$$

and

$$\bar{c}_i(G) = |\{A \subseteq V(G) : |A| = i, G[A] \notin \mathcal{C}\}|.$$

Clearly,

$$c_i(G) + \bar{c}_i(G) = \binom{n(G)}{i},$$

hence

$$\sum_{i=0}^{n(G)} (c_i(G) + \bar{c}_i(G)) X^i = (1 + X)^{n(G)}.$$

(ii) is similar, but we need that for a set of edges $A \subseteq E(G)$ the spanning subgraph $G\langle A \rangle = (V(G), A) \in \mathcal{D}$ iff $V(A), A \in \mathcal{D}$, where $V(A) = \{v \in V(G) : \text{there is } u \in V(G) \text{ with } (u, v) \in A\}$. \square

Proposition 4.7. Let \mathcal{C}_1 and \mathcal{C}_2 , \mathcal{D}_1 and \mathcal{D}_2 be graph properties such that \mathcal{C}_1 and \mathcal{C}_2 and \mathcal{D}_1 and \mathcal{D}_2 are pairwise d.p.-equivalent,

- (i) $\mathbf{P}_{\mathcal{C}_1}^{ind}(G; X)$ and $\mathbf{P}_{\mathcal{C}_2}^{ind}(G; X)$ are s.d.p.-equivalent;

- (ii) If, additionally, \mathcal{D}_1 and \mathcal{D}_2 are closed under the addition and removal of isolated vertices, then $\mathbf{P}_{\mathcal{D}_1}^{span}(G; X)$ and $\mathbf{P}_{\mathcal{D}_2}^{span}(G; X)$ are s.d.p.-equivalent;

Proof. We prove only (i), (ii) is proved analogously.

(i): We use Proposition 4.1. If $\mathcal{C}_1 = \mathcal{C}_2$, clearly, $\mathbf{P}_{\mathcal{C}_1}^{ind}(G; X) = \mathbf{P}_{\mathcal{C}_2}^{ind}(G; X)$, hence they are d.p.-equivalent. If $\mathcal{C}_1 = \bar{\mathcal{C}}_2$, we use Proposition 4.6 together with Proposition 2.2. But Proposition 4.6 depends on the $n(G)$, hence we get only that $\mathbf{P}_{\mathcal{C}_1}^{ind}(G; X) = \mathbf{P}_{\mathcal{C}_2}^{ind}(G; X)$, are d.p.-equivalent. \square

Let G_n be an indexed sequence of graphs such that the sequence of polynomials $X^{|V(G_n)|}$ is C-finite. This assumption is true for the sequences C_n , P_n , K_n , of cycles, paths and cliques, and all sequences G_n of graphs provided the function $|V(G_n)|$ is linear in n . In particular, it applies to Theorem 2.11. We shall now show that C-finiteness of the sequences of polynomials $\mathbf{P}_{\mathcal{C}}^{ind}(G_n; X)$ of Theorem 2.11 is a semantic property graph polynomials as generating functions. However, the particular form of the recurrence relation is not.

Theorem 4.8. *Let \mathcal{C}_1 and \mathcal{C}_2 , \mathcal{D}_1 and \mathcal{D}_2 be graph properties such that \mathcal{C}_1 and \mathcal{C}_2 and \mathcal{D}_1 and \mathcal{D}_2 are pairwise d.p.-equivalent, and let G_n be an indexed sequence of graphs. Furthermore, assume that the sequence of polynomials $X^{|V(G_n)|}$ is C-finite. Then*

- (i) $\mathbf{P}_{\mathcal{C}_1}^{ind}(G_n; \mathbf{X})$ is C-finite iff $\mathbf{P}_{\mathcal{C}_2}^{ind}(G_n; \mathbf{X})$ is C-finite.
- (ii) $\mathbf{P}_{\mathcal{D}_1}^{span}(G_n; \mathbf{X})$ is C-finite iff $\mathbf{P}_{\mathcal{D}_2}^{span}(G_n; \mathbf{X})$ is C-finite;

Proof. This follows in both cases from the fact that the sum and difference of two C-finite sequences is again C-finite together with Proposition 4.6. \square

4.3. Generalized chromatic polynomials

Recall from the introduction the definition of $\chi_{\mathcal{C}}(G; k)$ as the number of colorings of G with at most k colors such that each color class induces a graph in \mathcal{C} .

Theorem 4.9 (J. Makowsky and B. Zilber, cf. [KMZ11]). *$\chi_{\mathcal{C}}(G; k)$ is a polynomial in k for any graph property \mathcal{C} .*

In contrast to Proposition 4.6 the relationship between $\chi_{\mathcal{C}}(G; k)$ and $\chi_{\bar{\mathcal{C}}}(G; k)$ is not at all obvious.

Problem 4.10. *What can we say about $\chi_{\bar{\mathcal{C}}}(G; k)$ in terms of $\chi_{\mathcal{C}}(G; k)$?*

Proposition 4.11. *There are two classes \mathcal{C}_1 and \mathcal{C}_2 which are d.p.-equivalent but such that $\chi_{\mathcal{C}_1}$ and $\chi_{\mathcal{C}_2}$ are not d.p.-equivalent.*

Proof. Let \mathcal{C}_1 be all the disconnected graphs and let \mathcal{C}_2 be all the connected graphs. As they are complements of each other, they are d.p.-equivalent.

We compute for K_i :

$$\chi_{\mathcal{C}_1}(K_i; j) = 0, j \in \mathbb{N}^+$$

because there is no way to partition K_i into any number of disconnected parts. Hence $\chi_{\mathcal{C}_1}(K_i; X) = 0$.

$$\chi_{\mathcal{C}_2}(K_i; 2) = 2^i - 2$$

because every partition of K_i into two nonempty parts gives two connected graphs.

Therefore $\chi_{\mathcal{C}_2}$ distinguishes between cliques of different size, whereas $\chi_{\mathcal{C}_1}$ does not. \square

We note, however, that the analogue of Proposition 4.7 for generalized chromatic polynomials remains open.

4.4. d.p.-equivalence of graph polynomials

The converse of Theorem 4.7(i) and (ii) is not true:

Proposition 4.12. *There are graph properties \mathcal{C}_1 and \mathcal{C}_2 which are not d.p.-equivalent, but such that*

- (i) $\mathbf{P}_{\mathcal{C}_1}^{ind}(G; X)$ and $\mathbf{P}_{\mathcal{C}_2}^{ind}(G; X)$ are d.p.-equivalent.
- (ii) $\chi_{\mathcal{C}_1}(G; X)$ and $\chi_{\mathcal{C}_2}(G; X)$ are d.p.-equivalent.

Proof. For (i) Let $\mathcal{C}_1 = \{K_1\}$ and $\mathcal{C}_2 = \{K_2, E_2\}$ where E_n is the graph on n vertices and no edges. We compute:

$$\begin{aligned} \mathbf{P}_{\mathcal{C}_1}^{ind}(G; X) &= n(G) \cdot X \\ \mathbf{P}_{\mathcal{C}_2}^{ind}(G; X) &= \binom{n(G)}{2} \cdot X^2 \end{aligned}$$

For (ii) we choose $\mathcal{C}_1 = \{K_1\}$ as before, but $\mathcal{C}_2 = \{K_1, K_2, E_2\}$.

Claim 1:

$\chi_{\mathcal{C}_2}(G, X) \leq_{d.p.} n(G)$ Proof: Let G_1 and G_2 be two graphs with the same number of vertices. W.l.o.g. assume they have the same vertex set $V(G_1) = V(G_2) = V$. Now notice for every $f : V \rightarrow [k]$, f is a \mathcal{C}_2 -coloring of G_1 iff it is a f is a \mathcal{C}_2 -coloring of G_2 . Hence $\chi_{\mathcal{C}_2}(G_1, X) = \chi_{\mathcal{C}_2}(G_2, X)$ whenever G_1 and G_2 have the same number of vertices.

Claim 2:

$n(G) \leq_{d.p.} \chi_{\mathcal{C}_2}(G, X)$ Proof: First denote for every m , $n_{even}(m) = \prod_{i=0}^{m-1} \binom{2(m-i)}{2}$ and $n_{odd}(m) = \prod_{i=0}^{m-1} \binom{2(m-i)+1}{2}$. For every graph G , there is a natural number $m(G)$ such that $n(G) = 2m(G)$ or $n(G) = 2m(G) + 1$. If $n(G) = 2m(G)$, $\chi_{\mathcal{C}_2}(G, m(G)) = n_{even}(m(G))$. If $n(G) = 2m(G) + 1$, $\chi_{\mathcal{C}_2}(G, m(G)) = n_{odd}(m(G))$. Note $n_{odd}(r) > n_{even}(r)$ for every natural number r . The minimal natural number r such that $\chi(G, r) > 0$ is equal to $m(G)$. We get that the minimal r such that $\chi_{\mathcal{C}_2}(G, r) > 0$ determines $n(G)$. Hence $\chi_{\mathcal{C}_1} =_{d.p.} \chi_{\mathcal{C}_2}$. \square

We leave it to the reader to construct the corresponding counterexample for $\mathbf{P}_{\mathcal{D}}^{span}(G; X)$.

We cannot use Proposition 4.3 to show that there infinitely many d.p.-incomparable graph polynomials of the form $\mathbf{P}_{\mathcal{C}}^{ind}(G; X)$. However, we can construct explicitly infinitely many d.p.-incomparable graph polynomials of this form.

4.5. Many d.p.-inequivalent graph polynomials

For the rest of this section, let C_i be the undirected circle on i vertices, and C_i^* the graph which consists of a copy of C_{i-1} together with a new vertex v which is connected to exactly one of the vertices of C_{i-1} . Clearly, C_i and C_i^* are similar. Furthermore, let $\mathcal{C}_i = \{C_i\}$, and let G_i^k consist of the disjoint union of k -many copies of C_i , and let \hat{G}_i^k consist of the disjoint union of $k-1$ copies of C_i^* together with one copy of C_i . Again, \hat{G}_i^k and G_i^k are similar.

We compute:

Lemma 4.13.

$$\mathbf{P}_{\mathcal{C}_j}^{ind}(G_i^k; X) = \mathbf{P}_{\mathcal{C}_j}^{ind}(\hat{G}_i^k; X) = 0 \text{ for } i \neq j, i \neq j+1, \quad (\text{i})$$

$$\mathbf{P}_{\mathcal{C}_i}^{ind}(G_i^k; X) = k \cdot X^i \quad (\text{ii})$$

$$\mathbf{P}_{\mathcal{C}_i}^{ind}(\hat{G}_i^k; X) = X^i \quad (\text{iii})$$

Theorem 4.14. *For all i, j with $i \neq j$ and $i \neq j+1$ the polynomials $\mathbf{P}_{\mathcal{C}_i}^{ind}$ and $\mathbf{P}_{\mathcal{C}_j}^{ind}$ are d.p.-incomparable, hence there are infinitely many d.p.-inequivalent graph polynomials of the form $\mathbf{P}_{\mathcal{C}}^{ind}(G; X)$.*

Proof. Assume $i, j \geq 3$ with $i \neq j$ and $i \neq j+1$. We first prove $\mathbf{P}_{\mathcal{C}_i}^{ind} \not\prec_{d.p.} \mathbf{P}_{\mathcal{C}_j}^{ind}$ for $i \neq j$ and $i \neq j+1$.

We look at the graphs G_j^2 and \hat{G}_j^2 . $\mathbf{P}_{\mathcal{C}_j}^{ind}(G_j^2; X) = 2 \cdot X^j$ by Lemma 4.13(ii). $\mathbf{P}_{\mathcal{C}_j}^{ind}(\hat{G}_j^2; X) = X^j$ by Lemma 4.13(iii). Hence, $\mathbf{P}_{\mathcal{C}_j}^{ind}$ distinguishes between the two graphs G_j^2 and \hat{G}_j^2 . However, $\mathbf{P}_{\mathcal{C}_i}^{ind}(G_j^2; X) = \mathbf{P}_{\mathcal{C}_i}^{ind}(\hat{G}_j^2; X) = 0$, by Lemma 4.13(i). Hence, $\mathbf{P}_{\mathcal{C}_i}^{ind}$ does not distinguish between the two graphs.

To prove $\mathbf{P}_{\mathcal{C}_j}^{ind} \not\prec_{d.p.} \mathbf{P}_{\mathcal{C}_i}^{ind}$ for $j \neq i$ and $j \neq i+1$, we look at the graphs G_i^2 and \hat{G}_i^2 . In this case $\mathbf{P}_{\mathcal{C}_j}^{ind}$ does not distinguish between the two graphs G_i^2 and \hat{G}_i^2 , but $\mathbf{P}_{\mathcal{C}_i}^{ind}$ does. \square

Theorem 4.15. *There are infinitely many d.p.-inequivalent graph polynomials of the form $\mathbf{P}_{\mathcal{C}}^{span}(G; X)$.*

Proof. The proof mimics the proof of Theorem 4.14 with following changes: Instead of \mathcal{C}_i we use $\mathcal{D}_i = \{C_i \sqcup E_j : j \in \mathbb{N}\}$ and

$$\mathbf{P}_{\mathcal{D}_j}^{span}(G_i^k; X) = 0 \text{ for } i \neq j, i \neq j+1$$

$$\mathbf{P}_{\mathcal{D}_i}^{span}(G_i^k; X) = k \cdot X^i.$$

$$\mathbf{P}_{\mathcal{D}_j}^{span}(\hat{G}_i^k; X) = \begin{cases} 0 & i \neq j, i \neq j+1 \\ (k-1) \cdot X^j & i = j+1 \end{cases}$$

$$\mathbf{P}_{\mathcal{D}_i}^{span}(\hat{G}_i^k; X) = X^i.$$

□

Next we look at chromatic polynomials $\chi_i(G; X) = \chi_{c_i}(G; X)$. We use the following obvious lemma:

Lemma 4.16. (i) For $X = \lambda \in \mathbb{N}$:

$$\chi_i(G_i^k; \lambda) = \begin{cases} \lambda_{(k)} & \lambda \geq k \\ 0 & \text{else} \end{cases}$$

(ii)

$$\chi_j(G_i^k, \lambda) = 0$$

provided that $i \neq j$.

(iii)

$$\chi_j(\hat{G}_i^k, \lambda) = 0$$

provided that $k \geq 2$ or $k = 1, i \neq j$.

Theorem 4.17. For all $i \neq j$ the polynomials χ_i and χ_j are d.p.-incomparable, hence there are infinitely many d.p.-incomparable graph polynomials of the form χ_C .

Proof. $\chi_i \not\leq_{d.p.} \chi_j$:

We look at the graphs G_i^2 and \hat{G}_i^2 . By Lemma 4.16 χ_j does not distinguish between G_i^2 and \hat{G}_i^2 . However, χ_i distinguishes between them.

To show that $\chi_j \not\leq_{d.p.} \chi_i$, we look at the graphs G_j^2 and \hat{G}_j^2 . By Lemma 4.16 χ_i does not distinguish between G_j^1 and G_j^2 . However, χ_j does distinguish between them. □

4.6. Generating functions of a relation

If, instead of counting induced (spanning) subgraphs with a certain graph property \mathcal{C} (\mathcal{D}), we count r -ary relations with a property $\Phi(A)$, we get a generalization of both the generating functions of induced (spanning) subgraphs. Here the summation is defined by

$$\mathbf{P}_\Phi(G; X) = \sum_{A \subseteq E(G): \Phi(A)} X^{|A|}.$$

For example, the generating matching polynomial, defined as

$$m(G; X) = \sum_{A \subseteq E(G): \Phi_{\text{match}}(A)} X^{|A|}.$$

can be written as

$$m(G; X) = \sum_{A \subseteq E(G): G \langle A \rangle \in \mathcal{D}_{\text{match}}} X^{|A|}$$

with \mathcal{D}_{match} being the disjoint union of isolated vertices and isolated edges.

However, not every graph polynomial $\mathbf{P}_\Phi(G; X)$ can be written as a generating function of induced (spanning) subgraphs.

Consider the graph polynomial

$$DOM(G; X) = \sum_{A \subseteq V(G): \Phi_{dom}(A)} X^{|A|}$$

where $\Phi_{dom}(A)$ says that A is a dominating set of G .

We compute:

$$DOM(K_2; X) = 2X + X^2 \quad (5)$$

$$DOM(E_2; X) = X^2 \quad (6)$$

Theorem 4.18. (i) *There is no graph property \mathcal{C} such that*

$$DOM(G; X) = \mathbf{P}_{\mathcal{C}}^{ind}(G; X).$$

(ii) *There is no graph property \mathcal{D} such that*

$$DOM(G; X) = \mathbf{P}_{\mathcal{D}}^{span}(G; X).$$

Proof. (i): Assume, for contradiction, there is such a \mathcal{C} , and that $K_1 \in \mathcal{C}$. The coefficient of X in $\mathbf{P}_{\mathcal{C}}^{ind}(E_2; X)$ is 2 because $K_1 \in \mathcal{C}$. However, the coefficient of X in $DOM(E_2; X)$ is 0, by equation (6), a contradiction.

Now, assume $K_1 \notin \mathcal{C}$. The coefficient of X in $\mathbf{P}_{\mathcal{C}}^{ind}(K_2; X)$ is 0, because $K_1 \notin \mathcal{C}$. However, the coefficient of X in $DOM(K_2; X)$ is 2, by equation (5), another contradiction.

(ii): Assume, for contradiction, there is such a \mathcal{D} . The coefficient of X in $\mathbf{P}_{\mathcal{D}}^{span}(K_2; X)$ is ≤ 1 , because K_2 has only one edge. However, the coefficient of X in $DOM(K_2; X)$ is 2, by equation (5), a contradiction. \square

We can use Equation (5) also to show the following:

Theorem 4.19. *There is no graph property \mathcal{C} such that*

$$DOM(G; X) = \chi_{\mathcal{C}}(G; X).$$

Proof. First we note that $\chi_{\mathcal{C}}(G; 1) = 1$ iff $\chi_{\mathcal{C}}(G; 1) \neq 0$ iff $G \in \mathcal{C}$.

Assume that $K_2 \in \mathcal{C}$. Then we have, using Equation (5),

$$\chi_{\mathcal{C}}(K_2; 1) = 1 = DOM(K_2, 1) = 3,$$

a contradiction.

Assume that $K_2 \notin \mathcal{C}$. Then we have, using Equation (5),

$$\chi_{\mathcal{C}}(K_2; 1) = 0 = DOM(K_2, 1) = 3,$$

another contradiction. \square

4.7. Determinant polynomials

There are only two matrices associated with graphs which have been used to define graph polynomials: the adjacency matrix and the Laplacian. The two resulting determinant polynomials are d.p.-incomparable. It is conceivable to define other matrix presentations of graphs, and ask when they give rise to d.p.-equivalent determinant polynomials. The characterization and recognition problem in this case amounts to the question when the characteristic polynomial of a matrix is the the characteristic polynomial arising from a graph. However, in this paper we do not pursue this further.

4.8. Characterizing d.p.-equivalence for special classes of graph polynomials

Theorems 4.7 and Proposition 4.11 and Proposition 4.12 show that d.p.-equivalence of \mathcal{C} and \mathcal{C}_1 , respectively \mathcal{D} and \mathcal{D}_1 , is not enough to characterize d.p.-equivalence of generating functions or generalized chromatic polynomials defined by \mathcal{C} and \mathcal{D} . Sometimes d.p.-equivalence of graph properties only implies and s.d.p.-equivalence of the corresponding graph polynomials.

Problem 4.20. *Characterize d.p.-equivalence of graph polynomials arising from \mathcal{C} and \mathcal{D} as*

- (i) *generalized chromatic polynomials;*
- (ii) *generating functions of induced or spanning subgraphs;*
- (iii) *generating functions of relations.*

5. SOL-definability of graph polynomials

In this section we present the formalism of SOL-definable graph polynomials. The idea originated in [CMR01, Mak05] and was further developed in [Kot12]. It is the logical framework which includes all the examples of graph polynomials so far discussed in this paper. SOL-definable graph polynomials are given using a finite set of SOL-formulas ϕ_1, \dots, ϕ_s such that replacing all the formulas $\phi_i : i \leq s$ by logically equivalent formulas ψ_i the resulting polynomial remains the same.

Given a graph polynomial $\mathbf{P}(G; X)$ there are uncountably many d.p.-equivalent graph polynomials. However, there will be only countably many SOL-definable graph polynomials. To show that a certain property of a graph polynomial \mathcal{X} is not a semantic property, it suffices to find a s.d.p. (d.p.)-equivalent graph polynomial which does not have property \mathcal{X} . Allowing all s.d.p.-equivalent graph polynomials really misses the point.

We are really interested in semantic properties of graph polynomials in a specific prescribed form. Let \mathcal{F} be a family of graph polynomials, such as generalized chromatic polynomials, generating functions of induced or spanning subgraphs, or SOL-definable polynomials.

By restricting the graph polynomials under consideration to \mathcal{F} we say a property of a polynomial is semantically meaningful on \mathcal{F} if all graph polynomials $\mathbf{P} \in \mathcal{F}$ which are d.p.-equivalent (s.d.p.-equivalent) share this property.

Example 5.1. Let \mathcal{F} be the class of graph polynomials given by generating functions of induced subgraphs of property \mathcal{C} . Let $\mathbf{P}_{\mathcal{C}}^{\text{ind}}$ and $\mathbf{P}_{\mathcal{D}}^{\text{ind}}$ from Section 4. Let $\mathcal{D} = \bar{\mathcal{C}}$ be the complement of \mathcal{C} . By proposition 4.7 we have that $\mathbf{P}_{\mathcal{C}}^{\text{ind}}$ and $\mathbf{P}_{\mathcal{D}}^{\text{ind}}$ are s.d.p. equivalent. For $G \in \mathcal{C}$ we have that $\mathbf{P}_{\mathcal{C}}^{\text{ind}}(G; X)$ is monic, but by Proposition 4.6, $\mathbf{P}_{\mathcal{D}}^{\text{ind}}(G; X)$ is not necessarily monic.

Hence, monic is not a semantic property even for graph polynomials restricted to generating functions of induced subgraphs.

The framework of SOL-definable graph polynomials allows us to analyze the graph theoretic (=semantic) content of graph polynomials. To show that a property \mathcal{X} of graph polynomials is not a graph theoretic property it suffices to show:

For every SOL-definable graph polynomial $\mathbf{P} \in \mathcal{X}$ there is a d.p.- or s.d.p.-equivalent SOL-definable graph polynomial $\mathbf{Q} \notin \mathcal{X}$ which can be easily constructed from the definition of \mathbf{P} .

Usually, \mathbf{Q} is explicitly given from the formulas defining \mathbf{P} . In some cases \mathbf{Q} is obtained from \mathbf{P} using substitutions and, possibly, by adding a prefactor.

5.1. Second Order Logic SOL

We assume the reader is familiar with Second Order Logic. Let τ be a finite set of relation symbols, i.e., a purely relational vocabulary. We have individual variables v_i and relation variables $U_{\rho(i),i}$ of arity $\rho(i)$. The set of SOL(τ)-formulas is defined inductively. We define atomic formulas over τ and equality using the relation symbols from τ and the relation variables. We close under Boolean connectives, and existential and universal quantification over individual variables and relation variables. We denote SOL(τ)-formulas with Greek letters, ϕ, ψ, θ , possibly with indices.

Caveat: We use here m, n, k , here as summation indices and not as graph parameters.

Let $\mathbf{U} = (U_{\rho(1),1}, \dots, U_{\rho(n),n})$, $\mathbf{v} = (v_1, \dots, v_m)$. We write $\phi(\mathbf{U}, \mathbf{v})$ for the formula with the indicated free variables. Given a τ -structure \mathfrak{A} with universe A , relations $B_{\rho(i),i} \subseteq A^{\rho(i)}$ for $i \in [n]$ and $b_i \in A$ for $i \in [m]$, we put $\mathbf{B} = (B_{\rho(1),1}, \dots, B_{\rho(n),n})$, $\mathbf{b} = (b_1, \dots, b_m)$ and we write $\phi(\mathbf{B}, \mathbf{b})$ for the formula evaluated over \mathfrak{A} .

5.2. Definable graph polynomials

The notion of *definability of graph parameters and graph polynomials in SOL* was first introduced⁷ in [CMR01] and extensively studied in [GKM08, FKM11, KMZ11, Kot12, GKM12, KMR12, MKR13, KM14].

Let $\mathcal{R} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ a ring or field. Given a graph $G = (V, E)$ we define the set of interpreted terms $\text{SOLEVAL}(G)$ in $\mathcal{R}[\mathbf{X}]$ inductively.

⁷ In [CMR01] we also deal with definability in Monadic Second Order Logic MSOL, but in this paper this distinction is of no use. Note however, that the results also hold for MSOL.

- (i) Elements of $\mathcal{R}[\mathbf{X}]$ are in $\text{SOLEVAL}(G)$.
- (ii) $\text{SOLEVAL}(G)$ is closed under addition, subtraction and multiplication in $\mathcal{R}[\mathbf{X}]$.
- (iii) $\text{SOLEVAL}(G)$ is closed under substitution of indeterminates by elements of $\mathcal{R}[\mathbf{X}]$.
- (iv) (Small sums and products) If $t \in \text{SOLEVAL}(G)$, and $\phi(\mathbf{v})$ is a formula of $\text{SOL}(\tau)$ with individual variables v_1, \dots, v_ρ and non-displayed interpreted individual and relation parameters, then

$$\sum_{\mathbf{b} \in V^\rho: \phi(\mathbf{b})} t$$

and

$$\prod_{\mathbf{b} \in V^\rho: \phi(\mathbf{b})} t$$

are interpreted terms in $\text{SOLEVAL}(G)$.

- (v) (Large sums) If $t \in \text{SOLEVAL}(G)$, and $\phi(U)$ is a formula of $\text{SOL}(\tau)$ with relation variable U of arity ρ and non-displayed interpreted individual and relation parameters, then

$$\sum_{B \subseteq V^\rho: \phi(B)} t$$

is a term in $\text{SOLEVAL}(G)$.

- (vi) An expression $t \in \text{SOLEVAL}(G)$ defines for each graph uniformly a polynomial $t(G) \in \mathcal{R}[\mathbf{X}]$.
- (vii) A graph polynomial $P(G, \mathbf{X})$ is SOL -definable if there is an expression $t \in \text{SOLEVAL}(G)$ such that for each graph G we have $t(G) = \mathbf{P}(G; \mathbf{X})$.

We first give examples where we use *small*, i.e., polynomial sized sums and products:

Examples 5.2. (i) *The cardinality of V is MSOL-definable by*

$$\sum_{v \in V} 1$$

- (ii) *The number of connected components of a graph G , $k(G)$ is MSOL-definable by*

$$\sum_{C \subseteq V: \text{component}(C)} 1$$

where $\text{component}(C)$ says that C is a connected component. Although the sum ranges over subsets of V , it is small, because there are at most $|V|$ -many connected components.

- (iii) *The graph polynomial $X^{k(G)}$ is MSOL-definable by*

$$\prod_{c \in V: \text{first-in-comp}(c)} X$$

if we have a linear order in the vertices and $\text{first-in-comp}(c)$ says that c is a first element in a connected component.

Now we give examples with possibly *large*, i.e., exponential sized sums:

Examples 5.3. (iv) The number of cliques in a graph is MSOL-definable by

$$\sum_{C \subseteq V: \text{clique}(C)} 1$$

where $\text{clique}(C)$ says that C induces a complete graph.

(v) Similarly “the number of maximal cliques” is MSOL-definable by

$$\sum_{C \subseteq V: \text{maxclique}(C)} 1$$

where $\text{maxclique}(C)$ says that C induces a maximal complete graph.

(vi) The clique number of G , $\omega(G)$ is SOL-definable by

$$\sum_{C \subseteq V: \text{largest-clique}(C)} 1$$

where $\text{largest-clique}(C)$ says that C induces a maximal complete graph of largest size.

(vii) The clique polynomial of G is SOL-definable by

$$\sum_{C \subseteq V: \text{clique}(C)} \prod_{v \in C} X_v.$$

Now here are some prominent graph polynomials which are easily seen to be SOL-definable.

Examples 5.4. (i) Let $G = (V(G), E(G))$ be a loopless graph without multiple edges. Here we consider again at the characteristic polynomial of G , $P_{cc}(G; X)$, and the Laplacian polynomial, $P_L(G; X)$ from Example 2.4. To see that both $P_{cc}(G; X)$ and $P_L(G; X)$ are SOL-definable, we write them as a sum of two SOL-definable polynomials in distinct indeterminates X_1 and X_2 , and then put $X_1 = X$ and $X_2 = (-1) \cdot X$. Here we use that SOL-definable polynomials are closed under substitution by elements of the polynomial ring.

In other words, to express the determinant

$$P_B(G; X) = \det(X \cdot I - B)$$

of a matrix B dependent on G , we write

$$P_B(G; X_1, X_2) = P_B^{\text{even}}(X_1) + P_B^{\text{odd}}(X_2)$$

where $P_B^{\text{even}}(X_1)$ sums over all even permutations and where $P_B^{\text{odd}}(X_2)$ sums over all odd permutations and then put

$$P_B(G; X) = P_B^{\text{even}}(X) + P_B^{\text{odd}}((-1) \cdot X).$$

Now we can use this to show that $P_{cc}(G; X) = \det(X \cdot I - A_G)$ and $P_L(G; X) = \det(X \cdot I - L_G)$ are substitution instances of bivariate SOL-definable graph polynomials.

(ii) Let

$$a_i(G) = |\{U \subseteq V : (G, U) \models \phi(U) \text{ and } |U| = i\}|$$

be uniformly defined numeric graph parameters. Then

$$\sum_i a_i(G) X^i = \sum_{U: \phi(U)} X^{|U|}$$

is a the generic form of an SOL-definable graph polynomial.

(ii.a) If $\phi(U)$ says that U is a set of edges which form a matching, we get the matching generating polynomial $g(G; X)$.

(ii.b) If $\phi(U)$ says that U is a set of vertices which form an independent set, we get the independence polynomial $I(G; X)$.

(iii) The Potts model is the partition function

$$Z(G; X, Y) = \sum_{B \subseteq E(G)} X^{k[B]} Y^{|B|}$$

with $k[B]$ is the number of connected components of the spanning subgraph generated by B . $Z(G; X, Y)$ is SOL-definable if an order on the vertices is present, using the closure properties and the previous examples.

(iv) The chromatic polynomial $\chi(G; X)$ is SOL-definable, using closure under substitution and the fact that

$$\chi(G; X) = Z(G; X, -1)$$

Remark 5.5. Negative coefficients may occur in SOL-definable polynomials, however they do occur only as a result of substitution of negative numbers for indeterminates. In the above examples $Z(G; X, Y)$ has no negative coefficients, but $\chi(G; X) = Z(G; X, -1)$ does.

In general, to show that a graph polynomial is definable in SOL may be difficult. For instance, counting the number of planar induced subgraphs uses Kuratowski's or Wagner's characterization of planarity. We do not know a general method to show that a graph polynomial is not SOL-definable. To show that it is not MSOL-definable one can use the method of connection matrices, [KM14].

5.3. Normal form of SOL-definable graph polynomials

In [Kot12, KMZ11] the following normal form theorem was proved:

Theorem 5.6 (Normal Form Theorem). *Every SOL-definable multivariate graph polynomial $\mathbf{P}(G; \mathbf{X})$ can be written as*

$$\mathbf{P}(G; \mathbf{X}) = \sum_{A \subseteq V^{r_1}: \phi_1(A)} \dots \sum_{A \subseteq V^{r_s}: \phi_s(A)} \prod_{\mathbf{X}_1 \in A: \psi_1(A, \mathbf{X}_1)} X_1 \cdot \dots \cdot \prod_{\mathbf{X}_t \in A: \psi_t(A, \mathbf{X}_t)} X_t$$

with $\phi_i, i \leq s, \psi_j : j \leq t$ SOL-formulas.

This shows that every SOL-definable graph polynomial is a multiple generating function of several SOL-definable relations.

5.4. Semantically equivalent presentations of graph polynomials

Theorems 2.11 and 2.13 show that on certain SOL-definable graph properties two different graph polynomials have identical polynomials. In fact, if we assume SOL-definability, we can always achieve equality of the coefficients.

Theorem 5.7. *Assume \mathcal{K} is a SOL-definable graph class, and $\mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G; \mathbf{X})$ are SOL-definable and d.p.-equivalent on \mathcal{K} .*

- (i) *There is a SOL-definable graph polynomial $P'(G; \mathbf{X})$ which is d.p.-equivalent to $\mathbf{P}(G; \mathbf{X})$ and such that for all $G \in \mathcal{K}$ we have that*

$$\mathbf{P}'(G; \mathbf{X}) = \mathbf{Q}(G; \mathbf{X}).$$

- (ii) *If \mathcal{K} , $\mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G; \mathbf{X})$ are all computable (computable in exponential time), so is $P'(G; \mathbf{X})$.*

Proof. (i): We define

$$\mathbf{P}'(G; \mathbf{X}) = \begin{cases} \mathbf{Q}(G; \mathbf{X}) & \text{if } G \in \mathcal{K} \\ \mathbf{P}(G; \mathbf{X}) & \text{else .} \end{cases}$$

It is straightforward that $\mathbf{P}(G; \mathbf{X})$ and $P'(G; \mathbf{X})$ are d.p.-equivalent on all graphs, and satisfies the equality of the coefficients. To see that $P'(G; \mathbf{X})$ is SOL-definable we note that a case distinction given by a SOL-definable class \mathcal{K} is also SOL-definable.

(ii): The computability of $P'(G; \mathbf{X})$ and its complexity statement follow immediately from the computability and complexity assumptions. \square

5.5. Consistency and the recognition problem

Given a closed formula ϕ in a logical system consistency of ϕ asks whether there exists a structure \mathfrak{A} such that $\mathfrak{A} \models \phi$. If we consider $\hat{\phi}$ as a Boolean graph parameter, this can be expressed as asking whether there is a graph G such that $\hat{\phi}(G) = 1$.

The generalization of consistency for \mathcal{R} -valued graph parameters \mathbf{P} and a value $p \in \mathcal{R}$ asks whether there is a graph G such that $\mathbf{P}(G) = p$. In the case of the chromatic polynomial, H. Wilf in [Wil73] calls this the *recognition problem*. We assume that H. Wilf had a constructive answer in mind which was not only algorithmic but algebraic and qualitative. If the parameter is $\mathbf{P}(G; X) = X^{n(G)}$ the expected answer says: Given $p \in \mathbb{Z}[X]$ there is a graph G such that $\mathbf{P}(G; X) = p$ iff p is monic and consists of exactly one monomial X^n with exponent $n \geq 1$.

In finite model theory consistency is computationally (recursively) enumerable, and computable, provided there is a bound $b_\phi \in \mathbb{N}$ on the size of the smallest model of ϕ .

For a SOL-definable graph polynomial $\mathbf{P}(G; \mathbf{X})$ such a bound always exists, hence the recognition problem is decidable. To give a syntactic description of the form of $p = \mathbf{P}(G; X)$, provided G exists, is a difficult problem, and wide open even for the case of the characteristic or the chromatic polynomial. For a discussion of the recognition problem, cf. [KMR17a].

5.6. Closure properties

Here we look at closure properties of SOLEVAL under reducibilities via distinctive power. Clearly, SOLEVAL is not closed under the relation $\leq_{d.p.}$ and $\leq_{s.d.p.}$. If $\mathbf{P}(G; \mathbf{X})$ and $\mathbf{Q}(G; \mathbf{Y})$ are two graph polynomials with $\mathbf{P}(G; \mathbf{X}) \leq_{d.p.} \mathbf{Q}(G; \mathbf{Y})$, and one of them is in SOLEVAL, the other still may not be in SOLEVAL. However, we have defined SOLEVAL to be closed under substitutions of indeterminates by elements of the underlying polynomial ring. Hence we get:

- Proposition 5.8.** (i) If $\mathbf{P}(G; \mathbf{X}) \preceq_{subst} \mathbf{Q}(G; \mathbf{Y})$ and $\mathbf{Q}(G; \mathbf{Y}) \in \text{SOLEVAL}$, then $\mathbf{P}(G; \mathbf{Y}) \in \text{SOLEVAL}$.
(ii) Let $\mathbf{P}(G; X, \mathbf{Y}), R(G; \mathbf{Y}) \in \text{SOLEVAL}$ with indeterminates X and \mathbf{Y} . Then the result of substituting $R(G; \mathbf{Y})$ for X in $\mathbf{P}(G; X, \mathbf{Y})$, $\mathbf{P}(G; R(G; \mathbf{Y}), \mathbf{Y})$ is also in SOLEVAL.
(iii) If $\mathbf{P}(G; \mathbf{X}) \preceq_{prefactor} \mathbf{Q}(G; \mathbf{Y})$ using similarity functions in SOLEVAL, i.e., there are similarity functions $f(G; \mathbf{X}), g_1(G; \mathbf{X}), \dots, g_{m_2}(G; \mathbf{X}) \in \text{SOLEVAL}$ such that $\mathbf{P}(G; \mathbf{X}) = f(G; \mathbf{X}) \cdot \mathbf{Q}(G; g_1(G; \mathbf{X}), \dots, g_{m_2}(G; \mathbf{X}))$ and $\mathbf{Q}(G; \mathbf{Y}) \in \text{SOLEVAL}$, then $\mathbf{P}(G; \mathbf{Y}) \in \text{SOLEVAL}$.

Proof. (i) follows from the definition of SOLEVAL.

(ii) is shown by induction on the definition of $R(G; \mathbf{Y})$.

(iii) follows from (ii). □

6. On the location of zeros of graph polynomials

6.1. Roots of univariate graph polynomials

The literature on graph polynomials mostly got its inspiration from the successes in studying the chromatic polynomial and its many generalizations and the characteristic polynomial of graphs. In both cases the roots of graph polynomials are given much attention and are meaningful when these polynomials model physical reality.

A complex number $z \in \mathbb{C}$ is a root of a univariate graph polynomial $P(G; X)$ if there is a graph G such that $P(G; z) = 0$. It is customary to study the location of the roots of univariate graph polynomials. Prominent examples, besides the chromatic polynomial, the matching polynomial and the characteristic polynomial and its Laplacian version, are the independence polynomial, the domination polynomial and the vertex cover polynomial.

For a fixed univariate graph polynomial $P(G; X)$ typical statements about roots are:

- (i) For every G the roots of $P(G; X)$ are real. This is the univariate version of stability or Hurwitz stability for real polynomials. It is true for the characteristic and the matching polynomial, [CDS95, LP86]. Similarly, for every claw-free graph G the roots of the independence polynomial are real, [MC07]. Incidentally, by a classical theorem of I. Newton, if all the roots of a polynomial with positive coefficients are real, then its coefficients are unimodal.

- (ii) Assuming that all roots of $P(G; X)$ are real, the (second) largest root has an interesting combinatorial interpretation. This is true for the characteristic polynomial where the second largest eigenvalue is related to the Cheeger constant, [AM85, BH12, Chapter 4].
- (iii) The multiplicity of a certain value a as a root of $P(G; X)$ has an interesting interpretation. For example, the multiplicity of 0 as a root of the Laplacian polynomial is the number of connected components of G , [BH12, Chapter 1.3.7].
- (iv) For every G all real roots of $P(G; X)$ are positive (negative) or the only real root is 0. The real roots are positive in the case of the chromatic polynomial and the clique polynomial, and negative for the independence polynomial, [DKT05, HL94, BHN04, GS00, Hos07].
- (v) For every G the roots of $P(G; X)$ are contained in a disk of radius $\rho(d(G))$, where $d(G)$ is the maximal degree of the vertices of G . This is true for the characteristic polynomial and its Laplacian version, [BH12, Chapter 3]. This is also the case for the chromatic polynomial, [DKT05, Sok01], but the proof of this is far from trivial.
- (vi) For every G the roots of $P(G; X)$ are contained in a disk of constant radius. This is the case for the edge-cover polynomial, [CO11]. For the unit disk this is the univariate version of Schur-stability.
- (vii) The roots of $P(G; X)$ are dense in the complex plane. This is again true for the chromatic polynomial, the dominating polynomial and the independence polynomial, [DKT05, Sok04, BHN04, Hos07].

In [MRB14] we showed that the precise location of roots of univariate SOL-definable graph polynomials is not a graph theoretic (semantic) property of graphs. In the next section we investigate whether stability, the multivariate analog the location of zeros, of multivariate SOL-definable graph polynomials is a semantic property. A typical theorem from [MRB14] is the following.

Theorem 6.1 ([MRB14, Theorem 4.22]). *For every univariate graph polynomial $P(G; X) \in \text{SOLEVAL}$ there exists a univariate graph polynomial $Q(G; X)$ which is prefactor equivalent to $P(G; X)$ and the roots of $Q(G; X)$ are dense in \mathbb{C} . Furthermore, if $P(G; X) \in \text{SOLEVAL}$ so is $Q(G; X)$.*

To show this we use [MRB14, Lemma 4.21]:

Lemma 6.2. *There exist an SOL-definable univariate similarity polynomial $D_{\mathbb{C}}(G; X)$ of degree 48 such that all its roots are dense in \mathbb{C} .*

Proof of Theorem 6.1: We use Lemma 6.2 and put

$$Q(G; X) = D_{\mathbb{C}}(G; X) \cdot P(G; X)$$

and the fact that SOLEVAL is closed under products. □

6.2. Stable multivariate graph polynomials

A multivariate polynomial is *stable*⁸ if the imaginary part of its zeros is negative, and it is *Hurwitz-stable* if the real part of its zeros is negative. Analogously, it is *Schur-stable* if all its roots are in the open unit ball. Recently, stable and Hurwitz-stable polynomials have attracted the attention of combinatorial research. In [COSW04] the study of graph and matroid invariants and their various stability properties was initiated. The more recent paper [HM13] does the same for knot and link invariants. Due mainly to the recent work of J. Borcea and P. Brändén [BB⁺08], see also [Wag11], a very successful multivariate generalization of stability of polynomials has been developed. To quote from the abstract of [Vis13]:

Problems in many different areas of mathematics reduce to questions about the zeros of complex univariate and multivariate polynomials. Recently, several significant and seemingly unrelated results relevant to theoretical computer science have benefited from taking this route: they rely on showing, at some level, that a certain univariate or multivariate polynomial has no zeros in a region. This is achieved by inductively constructing the relevant polynomial via a sequence of operations which preserve the property of not having roots in the required region.

Further on, [Vis13] gives the following applications of stable polynomials to theoretical computer science: A new proof of the van der Waerden conjecture about the permanent of doubly stochastic matrices, [Gur06]; various applications to the traveling salesman problem, [Vis12], [Pem12]; applications to the Lee-Yang theorem in statistical physics that shows the lack of phase transition in the Ising model, [SS13], and more. [BBL09] discuss various sampling problems and show, among other things, that the generating polynomial of spanning trees of a graph is stable, see also [AGR16]. Let $m, n \in \mathbb{N}$ be indices. Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be $n + m$ indeterminates and $f(\mathbf{X}, \mathbf{Y}) \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]$. Let $\mathcal{H}_u = \{a \in \mathbb{C} : \Im(a) > 0\}$ and $\mathcal{H}_r = \{a \in \mathbb{C} : \Re(a) > 0\}$ be the upper, respectively right half-plane of \mathbb{C} .

Definitions 6.3. (i) *f is homogeneous if all its monomials have the same degree.*

⁸ Multivariate analogs of location of zeros of polynomials are the various halfplane properties aka stability properties.

In engineering and stability theory, a square matrix A is called stable matrix (or sometimes Hurwitz matrix) if every eigenvalue of A has strictly negative real part. These matrices were first studied in the landmark paper [Hur95] in 1895. The Hurwitz stability matrix plays a crucial part in control theory. A system is stable if its control matrix is a Hurwitz matrix. The negative real components of the eigenvalues of the matrix represent negative feedback. Similarly, a system is inherently unstable if any of the eigenvalues have positive real components, representing positive feedback. In the engineering literature, one also considers Schur-stable univariate polynomials, which are polynomials such that all their roots are in the open unit disk, see for example [WML94].

- (ii) f is multiaffine if each indeterminate occurs at most to the first power in f .
- (iii) $f \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]$ is stable if $f \equiv 0$ or, whenever $\mathbf{a} \in \mathcal{H}_u^{n+m}$, then $f(\mathbf{a}) \neq 0$. If additionally $f(\mathbf{X}) \in \mathbb{R}[\mathbf{X}, \mathbf{Y}]$, it is real stable.
- (iv) f is Hurwitz-stable if $f \equiv 0$ or, whenever $\mathbf{a} \in \mathcal{H}_r^{n+m}$, then $f(\mathbf{a}) \neq 0$.
- (v) f is stable with respect to \mathbf{X} if for every $\mathbf{b} \in \mathcal{H}^m$ either $f(\mathbf{X}, \mathbf{b}) \equiv 0$ or whenever $\mathbf{a} \in \mathcal{H}_u^n$ then $f(\mathbf{a}, \mathbf{b}) \neq 0$.
- (vi) Let \mathcal{K} be class of finite graphs. A graph polynomial $P(G; \mathbf{X})$ is stable on \mathcal{K} if for every graph $G \in \mathcal{K}$ the polynomial $P(G; \mathbf{X}) \in \mathbb{C}[\mathbf{X}]$ is stable.

Remark 6.4. If $f(\mathbf{X}, \mathbf{Y})$ is stable, it is stable with respect to \mathbf{X} , but not conversely.

Examples 6.5. (i) Univariate polynomials are stable iff they have only real roots.

- (ii) The characteristic polynomial P_{cc} and its Laplacian version P_L are stable because they have only real roots.
- (iii) Let $\text{Tree}(G; X) = \sum_{T \subseteq E(G)} \prod_{e \in T} X_e$, be the tree polynomial, where T ranges over all trees of $G = (V(G), E(G))$. $\text{Tree}(G; X)$ is Hurwitz-stable, [COSW04].
- (iv) Let $G = (V(G), E(G))$ be a graph and let $\mathbf{X}_E = (X_e : e \in E(G))$ be commutative indeterminates. Let S be a family of subsets of $E(G)$, i.e., $S \subset \wp(E(G))$ and let $P_S(G; \mathbf{X}_E) = \sum_{A \in S} \prod_{e \in A} X_e$. If S is the family of trees of $E(G)$ then $P_S(G; \mathbf{X}_E)$ is a multivariate version of the tree polynomial, which is also Hurwitz-stable, cf. [Sok05, Theorem 6.2].
- (v) In [COSW04, Question 1.3] it is asked for which S is the polynomial $P_S(G; \mathbf{X}_E)$ Hurwitz-stable. Actually they ask the corresponding question for matroids

$$M = (E(M), S(M)).$$

- (vi) In [HM13, Section 16] the stability of multivariate knot polynomials is studied.

6.3. Sufficient conditions for stability

The characteristic polynomial of a symmetric real matrix is stable. Stable polynomials are often determinant like in the following sense:

Theorem 6.6 (Criteria for Stability). Let $\mathbf{X} = (X_1, \dots, X_m)$ be indeterminates, and \mathcal{X} be the diagonal matrix of n indeterminates with $(\mathcal{X})_{i,i} = X_i$.

- (i) ([BB⁺08, Proposition 2.4]) For $i \in [m]$ let each A_i be a positive semi-definite Hermitian $(n \times n)$ -matrix and let B be Hermitian. Then

$$f(\mathbf{X}) = \det(X_1 A_1 + \dots + X_m A_m + B) \in \mathbb{R}[\mathbf{X}]$$

is stable.

- (ii) ([HV07, Theorem 2.2]) For $m = 2$ and $f(X_1, X_2) \in \mathbb{R}[X_1, X_2]$ we have $f(X_1, X_2)$ is stable iff there are Hermitian matrices A_1, A_2, B with A_1, A_2 positive semi-definite such that

$$f(X_1, X_2) = \det(X_1 A_1 + X_2 A_2 + B).$$

- (iii) ([Brä07, after Theorem 4.2]) If A is a Hermitian $(m \times m)$ matrix then the polynomials $\det(\mathcal{X} + A)$ and $\det(I + A \cdot \mathcal{X})$ are real stable.

Theorem 6.7 (Criteria for Hurwitz-stability). (i) ([WW09]) If $f(\mathbf{X}) \in \mathbb{R}[\mathbf{X}]$ is a real homogeneous then $f(\mathbf{X})$ is stable iff $f(\mathbf{X})$ is Hurwitz stable.
(ii) ([COSW04, Theorem 8.1]) Let A be a complex $(r \times m)$ -matrix, A^* be its Hermitian conjugate, then the polynomial in m -indeterminates

$$Q(\mathbf{X}) = \det(A\mathcal{X}A^*)$$

is multiaffine, homogeneous and Hurwitz-stable.

- (iii) ([Brä07, after Theorem 4.2]) If B is a skew-Hermitian $(n \times n)$ matrix then $\det(\mathcal{X} + B)$ and $\det(I + B \cdot \mathcal{X})$ are Hurwitz stable.
(iv) ([COSW04, Theorem 10.2]) Let A be a real $(r \times m)$ -matrix with non-negative entries. Then the polynomial in m -indeterminates

$$Q(\mathbf{X}) = \text{per}(AX) = \sum_{S \subseteq [m], |S|=r} \text{per}(A|_S) \prod_{i \in S} X_i$$

is Hurwitz-stable.

6.4. Making graph polynomials stable

We first consider graph polynomials with a fixed number of indeterminates m . Let $P(G; \mathbf{X})$ be a graph polynomial with integer coefficients and with SOL-definition

$$P(G; \mathbf{X}) = \sum_{\phi} \prod_{\psi_1} X_1 \cdots \prod_{\psi_m} X_m,$$

with coefficients $(c_{i_1, \dots, i_m} : i_j \leq d(G), j \in [m])$

$$P(G; \mathbf{X}) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} X_1^{i_1} X_2^{i_2} \cdots X_m^{i_m} \in \mathbb{N}[\mathbf{X}],$$

such that in each indeterminate the degree of $P(G, \mathbf{X})$ is less than $d(G)$. We put $M(G) = d(G)^m$ which serves as a bound on the number of relevant coefficients, some of which can be 0.

Theorem 6.8. *There is a stable graph polynomial $Q^s(G; Y, \mathbf{X})$ with integer coefficients such that*

- (i) *the coefficients of $Q^s(G)$ can be computed uniformly⁹ in polynomial time from the coefficients of $P(G)$;*

⁹ There is a polynomial time computable function $F : \mathbb{Z}[\mathbf{X}] \rightarrow \mathbb{Z}[Y, \mathbf{X}]$ such that for all graphs G we have $F(P(G; \mathbf{X})) = Q^s(G; Y, \mathbf{X})$.

- (ii) there is $a_0 \in \mathbb{N}$ such that $Q^s(G; a_0, \mathbf{X})$ is d.p.-equivalent to $P(G; \mathbf{X})$;
- (iii) $Q^s(G; Y, \mathbf{X})$ is SOL-definable and its SOL-definition can be computed uniformly in polynomial time from $\phi, \psi_1, \dots, \psi_m$.

Theorem 6.9. *If additionally, $P(G; \mathbf{X})$ has only non-negative coefficients, there is a Hurwitz-stable graph polynomial $Q^h(G; Y, \mathbf{X})$ with non-negative integer coefficients and one more indeterminate Y such that*

- (i) *the coefficients of $Q^h(G)$ can be computed uniformly in polynomial time from the coefficients of $P(G)$;*
- (ii) *there is $\mathbf{a} \in \mathbb{N}^{M-n}$ such that $Q^h(G; \mathbf{a}, \mathbf{X})$ is d.p.-equivalent to $P(G; \mathbf{X})$;*
- (iii) *$Q^h(G; Y, \mathbf{X})$ is SOL-definable and its SOL-definition can be computed uniformly in polynomial time from $\phi, \psi_1, \dots, \psi_m$.*

In [COSW04, Sok05] the authors also consider graph polynomials where the number of indeterminates depends on the graph $G = (V(G), E(G))$, as in Example 6.5(iv). We will not give the most general definition here, but restrict ourselves to the case the indeterminates X_e are labeled by the edges $E(G)$ of G . We put $m(G)$ to be the cardinality of $E(G)$.

Let $S(G; \mathbf{X})$ be a multiaffine graph polynomial with non-negative integer coefficients and with SOL-definition

$$S(G; \mathbf{X}) = \sum_{\phi(A)} \prod_{\psi_1(A, e)} X_e \cdots \prod_{\psi_m(A, e)} X_e,$$

and coefficients $(c_{i_1, \dots, i_m} : i_j \in \{0, 1\}, j \in [m(G)])$

$$S(G; \mathbf{X}) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} X_1^{i_1} X_2^{i_2} \cdots X_m^{i_m(G)} \in \mathbb{N}[\mathbf{X}],$$

such that in each indeterminate the degree of $S(G, \mathbf{X})$ is less than $d(G)$. We put $M(G) = 2^{m(G)}$ which serves as a bound on the number of relevant coefficients, some of which can be 0. Let $\mathbf{X}_G = (X_e : e \in E(G))$.

Theorem 6.10. *There are graph polynomials $T^s(G; \mathbf{X}_G)$ and $T^h(G; \mathbf{X}_G)$ with non-negative integer coefficients such that*

- (i) *$T^s(G; \mathbf{X}_G)$ is stable and $T^h(G; \mathbf{X}_G)$ is Hurwitz-stable;*
- (ii) *Both the coefficients of $T^s(G)$ and of $T^h(G)$ can be computed uniformly in polynomial time from the coefficients of $S(G)$;*
- (iii) *Both $T^s(G; \mathbf{X}_G)$ and $T^h(G; \mathbf{X}_G)$ are d.p.-equivalent to $S(G; \mathbf{X}_G)$;*
- (iv) *Both $T^s(G; \mathbf{X}_G)$ and $T^h(G; \mathbf{X}_G)$ are SOL-definable and its SOL-definition can be computed uniformly in polynomial time from $\phi, \psi_1, \dots, \psi_m$.*

6.5. Proofs

6.5.1. Proof of Theorem 6.8

We use Theorem 6.6(i). Let $\alpha : \mathbb{N}^m \rightarrow \mathbb{N}$ which maps $(i_1, \dots, i_m) \in \mathbb{N}^m$ into its position in the lexicographic order of \mathbb{N}^m . We relabel the coefficients of $P(G; \mathbf{X})$ such that $d_i = c_{i_1, \dots, i_m}$ with $\alpha(i_1, \dots, i_m) = i, i \in [M]$ and $M = d(G)^m$.

We put B to be the $(M \times M)$ diagonal matrix with $B_{i,i} = d_i \cdot Y_i$ and $A_1 = A_2 = \dots = A_m$ to be the $(M \times M)$ identity matrix. The identity matrix is both Hermitian and positive semi-definite. Furthermore, $B|_{Y=a} = B(a)$ being a diagonal matrix, is Hermitian for every $a \in \mathbb{C}$. Hence,

$$Q_a^s(G; a, \mathbf{X}) = \det(B(a) + \sum_{i=1}^M X_i \cdot A_i) = \prod_{i=1}^M (d_i + \sum_{i=1}^M X_i)$$

is stable for every $a \in \mathbb{C}$.

We have to verify (i)-(iii).

(i): All the matrices can be computed in polynomial time in $\mathbb{Z}[Y, \mathbf{X}]$.

(ii): We use Theorem 2.2: $Q^s \preceq_{d,p} P$ follows from (i). We have to show that there is $a_0 \in \mathbb{N}$ with $P \preceq_{d,p} Q_{a_0}^s$. The function α can be easily inverted. To recover the coefficients of $P(G)$ from the coefficients of $Q^s(G)$, we note that

$$\sum_{i=0}^M d_i(G) \cdot Y^i$$

is the coefficient of $(\sum_{\ell=1}^m X_\ell)^{M-1}$ of $Q^s(G; Y, \mathbf{X})$. This can be computed in polynomial time from the coefficients of Q^s . To find a_0 we let $a_0 \in \mathbb{N}$ be bigger than $1 + 2 \cdot |d_i(G)|$, as $d_i(G)$ could be negative. Now $\sum_{i=0}^M d_i(G) \cdot a_0^i$ can be viewed as a natural number written in base a_0 , and the digits $d_i(G)$ can be uniquely determined.

(iii): To prove that $Q^s(G; Y, \mathbf{X})$ is SOL-definable we need a few lemmas from [Kot12, FKM11, KMZ11].

The first lemma is part of the definition of SOL-definability.

Lemma 6.11. *Finite sums and products of SOL-definable polynomials are SOL-definable.*

Lemma 6.12. *Let $G_{<} = (V(G), E(G), < (G))$ be a graph with an ordering $< (G)$ on the vertices. Let $Q(G; \mathbf{X})$ be a graph polynomial with non-negative integer coefficients and with SOL-definition*

$$Q(G; \mathbf{X}) = \sum_{A \subseteq V^r: \phi(A)} \prod_{\mathbf{v}_1 \in A: \psi_1(A, \mathbf{v}_1)} X_1 \cdot \dots \cdot \prod_{\mathbf{v}_m \in A: \psi_m(A, \mathbf{v}_m)} X_m$$

with coefficients $(c_{i_1, \dots, i_m} : i_j \leq d(G), j \in [m])$

$$Q(G; \mathbf{X}) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} X_1^{i_1} X_2^{i_2} \dots X_m^{i_m} \in \mathbb{N}[\mathbf{X}].$$

such that in each indeterminate the degree of $P(G, \mathbf{X})$ is less than $d(G)$. Let $s(G)$ be such that $|V(G)|^{s(G)} \geq d(G)$ and extend the ordering $< (G)$ to the lexicographic ordering of $|V(G)|^{s(G)}$. For $\mathbf{v} \in V(G)^{s(G)}$ we define $\text{Init}(G; \mathbf{v})$ to be the set of predecessors of \mathbf{v} in this lexicographic ordering.

The coefficients c_{i_1, \dots, i_m} of $Q(G; \mathbf{X})$ are SOL-definable by

$$c(\mathbf{v}_1, \dots, \mathbf{v}_m) = \sum_{A \subseteq V^r} 1$$

where A ranges over all subsets satisfying $\phi(A)$ and for each $\ell \in [m]$ the set $\text{Init}(G; \mathbf{v}_\ell)$ is of the same size as i_ℓ and as

$$\{\mathbf{w}_\ell \in V^r : (V(G), E(G), <(G), A, \mathbf{w}_\ell) \models \phi(A) \wedge \psi(A, \mathbf{w}_\ell)\}$$

Proof. We only have to note that the equicardinality requirement is expressible in SOL. \square

Lemma 6.13. *The polynomial*

$$Q_a^s(G; a, \mathbf{X}) = \prod_{i=1}^M (d_i + \sum_{i=1}^M X_i) = \prod_{\mathbf{v}_1, \dots, \mathbf{v}_m} \left(c(\mathbf{v}_1, \dots, \mathbf{v}_m) + \sum_{i=1}^M X_i \right)$$

is SOL-definable.

6.5.2. Proof of Theorem 6.9

Now all the coefficients of $P(G; \mathbf{X})$ are non-negative. We want to use Theorem 6.7(i) together with Theorem 6.6(i). We repeat the proof of Theorem 6.8 with the following changes: Let D be the diagonal $(M \times M)$ -matrix of the coefficients, and Y a new indeterminate. Instead of $B(a)$ we use $D \cdot Y$ where Y is now a scalar. D is now a diagonal matrix with non-negative coefficients, so it is positive semi-definite. We put

$$Q(G; Y, \mathbf{X}) = \det(D \cdot Y + \sum_{i \in [m]} A_i \cdot X_i).$$

The resulting polynomial $Q(G; Y, \mathbf{X})$ is homogeneous and has integer coefficients. So we can apply Theorem 6.7(i) together with Theorem 6.6(i) to make to see that $Q(G; Y, \mathbf{X})$ is both stable and Hurwitz stable. In particular, for each $a \in \mathbb{N}$ $Q(G; a, \mathbf{X})$ is Hurwitz stable. To see that $Q(G; Y, \mathbf{X})$ is SOL-definable we again use $a \in \mathbb{N}$ large enough as in the proof of Theorem 6.9.

6.5.3. Proof of Theorem 6.10

The proof is the same as the proof of Theorem 6.9, where the number of indeterminates equals the number $m(G) = |E(G)|$.

7. Conclusions and open problems

In this paper we presented the logician's view of graph polynomials. This includes model theoretic reinterpretations of some of our previous work on graph polynomials, such as [Kot12, KMZ11, FKM11, MR13, MRB14, KMR17a]. We

systematically studied various notions of semantic equivalence of graph polynomials based on the notion of distinctive power. We were careful to set up this logical framework to be consistent with the way graph polynomials are compared in the graph theoretic literature. We also discussed various forms of graph polynomials, and unified all these under the framework of SOL-definable graph polynomials. Within this framework we also have a Normal Form Theorem 5.6.

In [MR13, MRB14] we initiated the study of semantic equivalence of univariate graph polynomials without focusing on definability or complexity. We showed there that the location of the roots are not a semantic property.

In this paper we have extended these studies to multivariate graph polynomials. We have also extended our framework threefold:

- (i) We have imposed computability restriction on our framework. To have a workable framework it does not suffice that the coefficients of a graph polynomial have to be computable from the graph, but that one needs to require that the inverse problem be decidable as well. This additional requirement was not used in [MKR13], where we were more concerned with complexity issues of evaluating graph polynomials.
- (ii) We have restricted our discussion to SOL-definable graph polynomials. This means that the d.p.-equivalent polynomial with stability properties has to be SOL-definable as well. In the univariate cases discussed in [MR13, MRB14] the additional definability requirement is not too difficult to be established. In the multivariate case, this is considerably more complicated.
- (iii) We have studied stability and Hurwitz-stability (aka the half-plane property) of multivariate graph polynomials. We have chosen this topic, because various graph polynomials arising from modeling natural phenomena turn out to be stable or Hurwitz-stable. Our study shows that these stability properties do not really reflect properties of the underlying graphs proper, but are the result of extraneous requirements arising from the particular modeling process of the natural phenomena in question.

Our work shows that to justify the study of the location of the zeroes of a graph polynomial, the particular choice of the coefficients of the graph polynomial has to be taken into account. If the only purpose of the graph polynomial is to encode purely graph theoretic properties, the location of its zeroes is irrelevant.

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